On the Homotopy Groups of the Suspended Quaternionic Projective Plane

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Abstract

In this paper, we computed the 2-localized and 3-localized homotopy groups $\pi_{r+k}(\Sigma^k\mathbb{H}P^2)$ where $r \leq 15$ and $k \geq 0$. And we gave the applications, including the classification of the 1-connected CW complexes having homology types of the suspended $\mathbb{H}P^3$ localized at 3 , and some decompositions of the self smashes.

Key words: Homotopy Groups; the Suspended Quaternionic Projective Plane

1 Introduction

Let $\mathbb{H}P^n$ denote the quaternionic projective space of dimension n. It is a quotient space of $S^{4n+3} \subseteq \mathbb{H}^{n+1}$ under the equivalence relation $x \sim \lambda x$ for any $x \in S^{4n+3}$ and $\lambda \in S^3 \subseteq \mathbb{H}$. $\mathbb{H}P^n$ is a CW complex having the type of $S^4 \cup e^8 \cup e^{12} \cup \cdots \cup e^{4n}$, and $\mathbb{H}P^{n-i}$ is its skeleton of dimension $4(n-i)$. $\mathbb{H}P^\infty$ is the union of all $\mathbb{H}P^n$ with the weak topology, having CW structure of type of $S^4 \cup e^8 \cup \cdots \cup e^{4k} \cup \cdots$ and $\mathbb{H}P^k$ is its skeleton of dimension $4k$. And there are famous Hopf fibrations $S^{4n+3} \to \mathbb{H}P^n$ $(1 \leq n \leq \infty)$ with fibre S^3. In this paper we mainly studied $\pi_{r+k}((\Sigma^k\mathbb{H}P^2)_{(p)})$ where $r \leq 15$ and $k \geq 0$, the homotopy groups of low dimensions of the suspended quaternionic projective plane localized at a prime p for $p = 2$ *or* 3. For $p \neq 2$ or 3, it is clear that $\Sigma\mathbb{H}P^2 \simeq S^5 \vee S^9$ localized at p, therefore, the homotopy groups $\pi_{r+k}(\Sigma^k\mathbb{H}P^2)$ ($\forall$ $r \leq 15$ and $k \geq 0$) localized at p , are already known . While for $r \leq 6$, the groups are the homotopy groups of spheres which have been known in the history. For $r \geq 7$, most of these groups in the stable range were studied by Juno Mukai (the second author) , and Aruns Liulevicius studied the groups $\pi^s_n(\mathbb{H}P^\infty)$ which have great relations with

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them. Juno Mukai mainly used the methods of the secondary compositions, and Aruns Liulevicius used the stable Adams sequence . For the groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ in the unstable range, generally speaking, with r growing, the situations are more mysterious. Some methods to compute the unstable homotopy groups are using the relative homotopy groups for $A \hookrightarrow X \to (X, A)$, and using the fibre sequence $\cdots \Omega\Sigma X \to F \to Y \cup CX \xrightarrow{pinch} \Sigma X$. In [2], B. Gray gave a method to decide the homotopy type of such F by his relative James construction . Furthermore , the unstable Adams sequence is a powerful tool to calculate the unstable homotopy groups.

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2 Preliminaries

2.0 Symbols

In this paper , by abuse of notation, when we take the p-localization skills, we always use the same symbols of the spaces, maps,homology groups and homotopy groups to denote the p-localized version of them, and $\widetilde{H}_*(-)$ denotes the mod p reduced homology .

The symbols of the generators of $\pi_{n+k}(S^n)$ we use are all given by H. Toda in [1], the only difference is that we write Toda's E as Σ, the familiar symbol nowadays, to denote the suspension functor.

And by abuse of notation, sometimes we use the same symbol to denote a map and its homotopy class, especially, for an element of a homotopy group, sometimes we use the same symbol to denote a map whose homotopy class is this element.

In this paper , all spaces and maps are in the category of pointed topological spaces and maps (i.e. continuous functions) preserving basepoint. And we always use $*$ to denote the basepoints and the constant maps mapping to the basepoints.

We often say the homotopy fibre (resp. cofibre) of a map $f : X \to Y$ as the fibre (resp. cofibre) of f for short, and we always use C_f to denote the maping cone (or say, cofibre)

of f . And the symbol N_f always denotes the space $X \times_f Y^I = \{(x, \varphi) \in X \times Y^I | f(x) = \varphi(1)\}$.

And for a CW complex $Z = Y \cup_g e^{m+1}$ where $Y = sk_m(Z)$, by abuse of notion , we often write the mapping cylinder M_g as M_Y to indicate that $M_Y \simeq Y$, although M_Y and M_Y/Y does not decided by Y but the attaching map g, one could see this kind of writing would make many benefits in this paper.

And we say the dual of the Steenord algebra as the Steenord algebra for short.

2.1 p-localization of a space

This section quotes the fundamental results of [9] without proofs.

Definition 2.1.1[9] Let p be a prime, $\mathbb{Z}_{(p)}$ denote p-local integers, we say that $X_{(p)}$ is a p-local space iff $\pi_*(X_{(p)})$ is p-local, i.e. $\pi_*(X_{(p)})$ is a $\mathbb{Z}_{(p)}$-module. We say that a map ℓ from a space X to a p-local space $X_{(p)}$ is a localization of X if it is universal for maps from X to local spaces , i.e. for any p-local space Y and any map $f : X \to Y$, there is a unique map $f_\ell : X_{(p)} \to Y$, such that $f = f_\ell \circ \ell$.

Proposition 2.1.2 [9] For any prime p, any simply connected space has a p-localization.

Proposition 2.1.3[9] If $\ell : X \to X_{(p)}$ is a p-localization of a simply connected space X, then ℓ induces isomorphisms,

$$\widetilde{H}_*(X_{(p)};\mathbb{Z}) \approx \widetilde{H}_*(X;\mathbb{Z}) \otimes \mathbb{Z}_{(p)} ,$$
$$\pi_*(X_{(p)}) \approx \pi_*(X) \otimes \mathbb{Z}_{(p)} .$$

2.2 the Extension Problem of $\mathbb{Z}_{(p)}$ Modules Extended by $\mathbb{Z}/p^r$

In this part, p always denotes a prime and $r \geq 1$. And for a $\mathbb{Z}_{(p)}$ module M and its nonempty subset S, the symbol $\langle S \rangle := \bigcap_{S \subseteq S' \leq M} S'$, the smallest $\mathbb{Z}_{(p)}$ submodule of M generated by S, and if

$$S = \{a_1, a_2, \cdots a_n\} \text{ is finite,}$$

$\langle \{a_1, a_2, \cdots a_n\} \rangle$ is written as $\langle a_1, a_2, \cdots a_n \rangle$ for short.

Lemma 2.2.1 Let $0 \to A \xrightarrow{i} B \xrightarrow{q} \mathbb{Z}/p^r \to 0$ be an exact sequence of $\mathbb{Z}_{(p)}$ modules for some homomorphisms i, q, then such $\mathbb{Z}_{(p)}$ modules B are given by

$$B \approx (A \oplus \mathbb{Z}_{(p)}) / \langle \ \{ \ (\zeta(x), -p^r x) \ \mid \ x \in \mathbb{Z}_{(p)} \ \} \rangle,$$

where $\zeta \in Hom_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}, A)$, and if the homological class $[\zeta] \in Ext^1_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^r, A)$ runs over $Ext^1_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^r, A)$, the fomula gives all such $\mathbb{Z}_{(p)}$ modules B satisfying the exact sequence above.

Proof

We only need to notice that

$$\cdots \to 0 \to \mathbb{Z}_{(p)} \xrightarrow{\times p^r} \mathbb{Z}_{(p)} \xrightarrow{proj.} \mathbb{Z}/p^r \to 0$$

is a projective resolution of $\mathbb{Z}/p^r$ as $\mathbb{Z}_{(p)}$ module , then our Lemma is followed by *theorem* 7.30 *of* [9], *page* 425 immediately . $\square$

Proposition 2.2.2 Let $m, n \in \mathbb{Z}_+$ and $t = min\{m, n\}$, if the sequence

$$0 \to \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m \longrightarrow B \longrightarrow \mathbb{Z}/p^n \to 0$$

of $\mathbb{Z}_{(p)}$ modules is exact, then all of such $\mathbb{Z}_{(p)}$ modules B are given by

$$B \approx \mathbb{Z}/p^m \oplus \mathbb{Z}/p^i \oplus \mathbb{Z}_{(p)}, \quad 0 \le i \le t-1$$
$$\text{or } \mathbb{Z}/p^{m+i-j} \oplus \mathbb{Z}/p^j \oplus \mathbb{Z}_{(p)}, \quad 0 \le j \le i \le n \text{ and } j \le t.$$

Proof

For the projective resolution of $\mathbb{Z}/p^n$ as $\mathbb{Z}_{(p)}$ module

$$\cdots \longrightarrow 0 \xrightarrow{d_2} \mathbb{Z}_{(p)} \xrightarrow[\times p^n]{d_1} \mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}/p^n \to 0$$

We have the chain ,

$$0 \to Hom(\mathbb{Z}/p^n, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \to Hom(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \xrightarrow{d_1^*} Hom(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \xrightarrow{d_2^*} 0.$$

Then $Ext^1_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^n, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) = Ker(d_2^*)/Im(d_1^*) = Hom(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m)/Im(d_1^*) \approx \mathbb{Z}/p^n \oplus \mathbb{Z}/p^t$, $t = min\{m, n\}$. Let $\varepsilon_1', \varepsilon_2' \in Hom(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m)$, where $\varepsilon_1'(1) = (1, 0)$ and $\varepsilon_2'(1) = (0, 1)$. And let $\varepsilon_1 = \varepsilon_1' + Im(d_1^*)$, $\varepsilon_2 = \varepsilon_2' + Im(d_1^*)$. Therefore, $Ext^1_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^n, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) = \mathbb{Z}/p^n\{\varepsilon_1\} \oplus \mathbb{Z}/p^t\{\varepsilon_2\}$. We notice that

$$((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}/p^n) \ / \langle\{ \ (\ (\lambda p^i \varepsilon_1'(x), \mu p^j \varepsilon_2'(x)), -p^n x) \ \mid \ x \in \mathbb{Z}_{(p)}\}\rangle$$
$$= ((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}/p^n) \ / \langle\{ \ (\ (p^i \varepsilon_1'(x), p^j \varepsilon_2'(x)), -p^n x) \ \mid \ x \in \mathbb{Z}_{(p)}\}\rangle$$

if $gcd(\lambda, p) = gcd(\mu, p) = 1$, $(\lambda, \mu \in \mathbb{Z}$). Then by Lemma 2.2-1, all of the $\mathbb{Z}_{(p)}$ modules B are given by

$$B \approx ((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}/p^n)/\langle\{ \ (\ (p^i \varepsilon_1'(x), p^j \varepsilon_2'(x)), -p^n x) \ \mid \ x \in \mathbb{Z}_{(p)}\}\rangle,$$

where $0 \le i \le n$ and $0 \le j \le t$.

If $i < j$,

$$((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}/p^n)/\langle\{ \ (\ (p^i \varepsilon_1'(x), p^j \varepsilon_2'(x)), -p^n x) \ \mid \ x \in \mathbb{Z}_{(p)}\}\rangle$$
$$\approx \frac{\mathbb{Z}_{(p)}\{a,b,c\}}{\langle p^m b, \ p^i a + p^j b - p^n c\rangle} = \frac{\mathbb{Z}_{(p)}\{a,b,c\}}{\langle p^m b, \ p^i(a + p^{j-i} b - p^{n-i} c)\rangle} = \frac{\mathbb{Z}_{(p)}\{a + p^{j-i} b - p^{n-i} c, \ b, \ c\}}{\langle p^m b, \ p^i(a + p^{j-i} b - p^{n-i} c)\rangle} \approx \mathbb{Z}/p^m \oplus \mathbb{Z}/p^i \oplus \mathbb{Z}_{(p)},$$

in this case, $0 \leq i < j \leq t$, so $0 \leq i \leq t-1$.

If $i \geq j$, let $b' = p^{i-j}a + b - p^{n-j}c$ and $k = i - j$, then

$$
\begin{aligned}
&((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}/p^n)/\langle \{ \ (\ (p^i\varepsilon_1'(x), p^j\varepsilon_2'(x)), -p^n x) \ \mid \ x \in \mathbb{Z}_{(p)} \} \rangle \\
&\approx \frac{\mathbb{Z}_{(p)}\{a,b,c\}}{\langle p^m b,\ p^i a + p^j b - p^n c\rangle} = \frac{\mathbb{Z}_{(p)}\{a,b,c\}}{\langle p^m b,\ p^j(p^{i-j}a + b - p^{n-j}c)\rangle} = \frac{\mathbb{Z}_{(p)}\{a,\ p^{i-j}a + b - p^{n-j}c,\ c\}}{\langle p^m b,\ p^j(p^{i-j}a + b - p^{n-j}c)\rangle} = \frac{\mathbb{Z}_{(p)}\{a,\ b',\ c\}}{\langle p^{m+n-j}c - p^{m+i-j}a,\ \ p^j b'\rangle} \\
&= \frac{\mathbb{Z}_{(p)}\{a,\ b',\ c\}}{\langle p^{m+i-j}(p^{n-i}c - a),\ \ p^j b'\rangle} = \frac{\mathbb{Z}_{(p)}\{p^{n-i}c - a,\ b',\ c\}}{\langle p^{m+i-j}(p^{n-i}c - a),\ \ p^j b'\rangle} \approx \mathbb{Z}/p^{m+i-j} \oplus \mathbb{Z}/p^j \oplus \mathbb{Z}_{(p)} \\
&\qquad = \mathbb{Z}/p^{m+k} \oplus \mathbb{Z}/p^j \oplus \mathbb{Z}_{(p)},
\end{aligned}
$$

in this case, $0 \leq j \leq i \leq n$ and $j \leq t$. □

2.3 CW Structures of the fibres and Cofibres

A weak homotopy equivalence between CW complexes is a homotopy equivalence, and a homology equivalence between simply connected CW complexes is a homotopy equivalence. Sometimes we hope to have a homotopy equivalence between spaces while they have been weak homotopy equivalent , these encourage us to observe the CW structures of spaces. About these, we have the following proposition.

Proposition 2.3.1[10]

(**i**) Let $f : X \to Y$ be a fibration where X and Y both have homotopy types of CW complexes and X is path connected, then the fibre of f also has a homotopy type of CW complex.

(**ii**) Let $f : X \to Y$ be a cellular map between CW complexs, then the mapping cylinder and the mapping cone of f are both CW complexes.

(**iii**) A space X has the type of a CW complex iff X is dominated by a CW-complex.

2.4 Some Foundmental Facts on Homotopy Groups

Proposition 2.4.1[11] The homotopy groups of a simply connected space are finitely generated ablian groups if its homology groups with $\mathbb{Z}$ coefficients are finitely generated ablian groups.

Therefore, the homotopy groups $\pi_i(\Sigma^j \mathbb{H}P^2)$ are all finitely generated ablian groups, and $\pi_i((\Sigma^j \mathbb{H}P^2)_{(p)})$ are all finitely generated modules over $\mathbb{Z}_{(p)}$. (p: prime)

Proposition 2.4.2[12] Let X be a CW complex, then for any $m \geq 1$ and any $r \geq 1$, the inclusion map $i : sk_{r+m}(X) \hookrightarrow X$ induces an isomorphism,

$$i_* : \pi_r(sk_{r+m}(X)) \xrightarrow{\approx} \pi_r(X) .$$

Proposition 2.4.3[12] (Freudenthal suspension theorem) Let X be an $(n-1)$- connected CW complex, then the suspension homomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$. We say $\pi_i(X)$ is in the stable range if $i < 2n-1$.

Remark 2.4.4 Let X be an $(n-1)$- connected CW complex, then $\pi_{d+k}(\Sigma^k X)$ is in the stable range$\Leftrightarrow$ $d+k < 2(n+k)-1 \Leftrightarrow k \geq d-2n+2$. Hence $\pi_{d+k}(\Sigma^k \mathbb{H}P^m)$ $(1 \leq m \leq \infty)$ is in the stable range $\Leftrightarrow k \geq d-6$.

Proposition 2.4.5[13] For any i, there are isomorphisms:

$\pi_i(\mathbb{H}P^2) \approx \pi_{i-1}(S^3) \oplus \pi_i(S^{11})$, $\pi_i(\mathbb{H}P^\infty) \approx \pi_{i-1}(S^3)$, induced by the Hopf fibre sequences $S^3 \hookrightarrow S^{4n+3} \to \mathbb{H}P^n$ by taking $n = 2, \infty$.

Lemma 2.4.6 (*Page 176 of* [1], *Formula 13.5, H.Toda*) After localization at 3, the 2-fold suspension homomorphisms

$$\pi_i(S^{2m-1}) \xrightarrow{\Sigma^2} \pi_{i+2}(S^{2m+1})$$

are isomorphisms if $i < 6m-3$.

Lemma 2.4.7 After localization at 3, the suspension homomorphisms

$$\Sigma : \ \pi_{12+k}(S^{5+k}) \longrightarrow \pi_{13+k}(S^{6+k}) \ \ (k \geq 0)$$

are all isomorphisms.

Proof

After localization at 3, it is known $\pi_{12+k}(S^{5+k}) \approx \mathbb{Z}/3$ $(k \geq 0)$. By lemma 2.1.4, the two compositions

$$\pi_{12}(S^5) \xrightarrow{\Sigma} \pi_{13}(S^6) \xrightarrow{\Sigma} \pi_{14}(S^7) ,$$
$$\pi_{14}(S^7) \xrightarrow{\Sigma} \pi_{15}(S^8) \xrightarrow{\Sigma} \pi_{16}(S^9)$$

are both isomorphisms, and $\pi_{16+j}(S^{9+j}) \xrightarrow{\Sigma} \pi_{17+j}(S^{10+j})$ $(j \geq 0)$ are isomorphisms because they are in the stable range. We notice that a self-homomorphism of $\mathbb{Z}/3$ is an isomor-

phism $\Leftrightarrow$ it is onto $\Leftrightarrow$ it is into. Hence, the suspension homomorphisms $\pi_{12+k}(S^{5+k}) \xrightarrow{\Sigma} \pi_{13+k}(S^{6+k})$ $(k \geq 0)$ are all isomorphisms. $\square$

2.5 A Lemma on Having the Same Homotopy Type

Lemma 2.5.1 After localization at p, suppose X and Y are both CW complexs, and $[f], [g] \in [\Sigma^2 X, Y]$ satisfying , $[f] = \lambda[g]$, $gcd(\lambda,\ p) = 1$. Then the cofibre s of f and g have the same homotopy type. (Hence, all cofibre s of $[\Sigma^2 X, Y]$'s generators have the same homotopy type if $[\Sigma^2 X, Y] \approx \mathbb{Z}/p^r$).

Proof.

Let $[\lambda]$ denote the self map of $\Sigma^2 X$ of degree $[\lambda]$, that is $[\lambda] : S^1 \wedge \Sigma X \to S^1 \wedge \Sigma X$, $z \wedge t \mapsto z^\lambda \wedge t$ $(S^1 \subseteq \mathbb{C})$. We observe the natural transformation of the cofibrations, we have a map $\varphi : C_f \to C_g$ and homotopy-commutative squares in following,

$$\begin{tikzcd} \Sigma^2 X \arrow[r, "f"] \arrow[d, "{[\lambda]}"'] & Y \arrow[r] \arrow[d, "id"'] & C_f \arrow[r] \arrow[d, "\varphi"'] & \Sigma^3 X \arrow[r] \arrow[d] & \cdots \\ \Sigma^2 X \arrow[r, "g"] & Y \arrow[r] & C_g \arrow[r] & \Sigma^3 X \arrow[r] & \cdots \end{tikzcd}$$

It induces commutative diagram of mod p reduced homology groups with exact rows. Since $gcd(\lambda, p) = 1$, so $[\lambda]_*$ are all isomorphisms. Then $\varphi_* : \widetilde{H}_i(C_f) \to \widetilde{H}_i(C_g)$ must be an isomorphism for each i by the Five Lemma. Therefore, according to the p-local Whitehead theorem, $C_f \simeq C_g$. $\square$

2.6 the Homotopy fibre of the Pinch Map from Mapping Cone

Suppose (A, a_0) is a closed subspace of (X, a_0), we will use a space $J(X, A)$ named the relative James construction for the pair of spaces. It was defined in [2] by professor Brayton Gray who wrote it as $(X, A)_\infty$, while we write it as $J(X, A)$ to parallel with the familiar symbol nowadays $J(X)$, the (absolute) James construction ; and we write the r-th filtration of $J(X, A)$ as $J_r(X, A)$ to parallel with the familiar symbol nowadays $J_r(X)$,while professor Gray wrote it as $(X, A)_r$. The relative James construction $J(X, A)$ is a CW complex if X is a CW complex and A is its subcomplex, and $J(X, A)$ is homotopy equivalent to the homotopy fibre of the pinch map $X \cup CA \to \Sigma A$ in certain cases.

And in next part, we use the symbol $\overset{w}{\simeq}$ to denote the weak homotopy equivalence, and we regard the 0-fold smash product $A^{\wedge 0}$ as S^0.

Proposition 2.6.1 (*B. Gray*, *Theorem* 4.2 *of* [2]) Suppose (A, a_0) is a closed subspace of (X, a_0) , $A \overset{i}{\hookrightarrow} X$ is a cofibration where i is the inclusion , and F is the fibre of $X \cup_i CA \xrightarrow{pinch} \Sigma A$, then $F \overset{w}{\simeq} J(X, A)$. Of course $F \simeq J(X, A)$ if F and $J(X, A)$ both have homotopy types of CW complexes .

Proposition 2.6.2 (*B. Gray*, *Theorem* 5.3 *of* [2]) Suppose (A, a_0) is a closed subspace of (X, a_0) , and a_0 is non-degenerate in both A and X , namely, $\{a_0\}$ is closed in both A and X as well as the inclusions $\{a_0\} \hookrightarrow A$ and $\{a_0\} \hookrightarrow X$ are cofibrations . Then there exists homotopy equivalence:

$$\Sigma J(X, A) \simeq \bigvee_{n \geq 0} (\Sigma X \wedge A^{\wedge n}) .$$

Remark 2.6.3 ([10], *Page* 252) The basepoint of any pointed CW complex is non-degenerate.

Proposition 2.6.4 (B. Gray,Corollary 5.8 of [2]) Suppose (A, a_0) is a closed subspace of (X, a_0) and $i : A \hookrightarrow X$ is the inclusion, $X = \Sigma X', A = \Sigma A'$. Then

$$J_2(X, A) = X \cup_\gamma C(X \wedge A')$$

where $\gamma = [1_X, i]$, the (generalized) Whitehead product.

Theorem 2.6.5 Let F_k be the fibre of $\Sigma^k \mathbb{H}P^2 \xrightarrow{pinch} S^{8+k}$, then $sk_{10+2k}(F_k) = S^{4+k}$. What's more, after localization at 2 or 3 , $sk_{20}(F_1) = S^5 \vee S^{13}$, $sk_{26}(F_3) = S^7 \vee S^{17}$.

Proof

Let $\mathbb{H}P^2 = S^4 \cup_g e^8$, $M_{\Sigma^k g}$ be the mapping cylinder of $\Sigma^k g : S^{7+k} \to S^{4+k}$, i be the inclusion $S^{7+k} \hookrightarrow M_{\Sigma^k g}$. By lemma 123, $M_{\Sigma^k g}$, $M_{\Sigma^k g} \cup_i CS^{7+k}$ and F_k are all CW complexes. Then $J(M_{\Sigma^k g}, S^{7+k})$ is a CW complex , and the basepoint $*$ is non-degenerate in both S^{7+k} and $M_{\Sigma^k g}$. We assert that F_k is the fibre of the pinch map $p'_k : M_{\Sigma^k g} \to S^{8+k}$ up to homotopy. In fact, by naturality , we have the homotopy-commutative diagrams with rows cofibrations,

$$\begin{tikzcd}
S^{7+k} \arrow[r, "\Sigma^k g"] \arrow[d, "id"'] & S^{4+k} \arrow[r] \arrow[d, "\subseteq"', "\simeq"] & \Sigma^k \mathbb{H}P^2 \arrow[r, "p_k"] \arrow[d, "\varphi"] & S^{8+k} \arrow[r] \arrow[d, "id"'] & \cdots \\
S^{7+k} \arrow[r, "i"] & M_{\Sigma^k g} \arrow[r] & M_{\Sigma^k g} \cup_i CS^{7+k} \arrow[r, "p'_k"] & S^{8+k} \arrow[r] & \cdots
\end{tikzcd}$$

Therefore, the map φ is a homotopy equivalence, successively, p_k and p'_k have the same fibre up to homotopy. We knew that $J(M_{\Sigma^k g}, S^{7+k})$ is a CW complex ,and by Proposition 2.3.1, F_k is a CW complex . According to Proposition 2.6.2, $F_k \simeq J(M_{\Sigma^k g}, S^{7+k})$. According to Proposition 2.1-6 and Proposition 2.1-7 ,we have

$$\Sigma F_k \simeq \bigvee_{n\geq 0}(\Sigma M_{\Sigma^k g} \wedge (S^{7+k})^{\wedge n}) \simeq \bigvee_{n\geq 0}(\Sigma S^{4+k} \wedge (S^{n(7+k)})) = \bigvee_{n\geq 0} S^{5+k+n(7+k)}. \tag{1}$$

Hence $\widetilde{H}_*(\Sigma F_k;\mathbb{Z}) = \bigoplus_{n\geq 0}\mathbb{Z}\{x_{5+k+n(7+k)}\}$, $(|x_{5+k+n(7+k)}| = 5 + k + n(7 + k))$.
Thus $\widetilde{H}_*(F_k;\mathbb{Z}) = \bigoplus_{n\geq 0}\mathbb{Z}\{y_{4+k+n(7+k)}\} = \mathbb{Z}\{y_{4+k}, y_{11+2k}, y_{18+3k}, \cdots\}$,
$(|y_{4+k+n(7+k)}| = 4 + k + n(7 + k))$. This gives $sk_{10+2k}(F_k) = S^{4+k}$.

Now, suppose all cases are after localization at 2. Since $\pi_{13}(S^6) = \mathbb{Z}/2$, $\pi_{17}(S^8) = (\mathbb{Z}/2)^5$, we could set $sk_{20}(F_1) = S^5 \cup_f e^{13}$ and $sk_{26}(F_3) = S^7 \cup_{f'} e^{17}$ satisfying $\Sigma f \simeq *$, $\Sigma g \simeq *$. By (5.14) of [1], $Page$ 48, $\Sigma : \pi_{12}(S^5) \to \pi_{13}(S^6)$ is into, and by (5.15) of [1], $Page$ 50, $\Sigma : \pi_{16}(S^7) \to \pi_{17}(S^8)$ is into. Therefore, $f \simeq *$, $f' \simeq *$.

Next, suppose all cases are after localization at 3. Since $\pi_{13}(S^6) = \mathbb{Z}/3$, $\pi_{16}(S^7) = 0$, we could set $sk_{20}(F_1) = S^5 \cup_f e^{13}$ satisfying $\Sigma f \simeq *$, and $sk_{26}(F_3) = S^7 \vee S^{17}$ By Lemma 2.1-5, $\Sigma^2 : \pi_{12}(S^5) \to \pi_{14}(S^7)$ is an isomorphism, it is clearly that $\Sigma^2 f \simeq *$, so $f \simeq *$. $\square$

Remark 2.6.6 From now on, the symbols $i_k, p_k, \partial_k, \varphi_{k+t}, w_{k+t}$ and θ_{k+t} always denote the maps associated with the homotopy-commutative diagram with the fibre sequences as rows.

$$\begin{tikzcd}
\cdots \to \Omega S^{8+k} \arrow[r, "\partial_k"] \arrow[d] & F_k \arrow[r, "i_k"] \arrow[d, "\varphi_{k+t}"] & \Sigma^k \mathbb{H}P^2 \arrow[r, "p_k"] \arrow[d, "w_{k+t}=\Omega^t\Sigma^t"] & S^{8+k} \arrow[d, "\theta_{k+t}=\Omega^t\Sigma^t"] \\
\cdots \to \Omega^{t+1} S^{8+k+t} \arrow[r, "\Omega^t\partial_{k+t}"] & \Omega^t F_{k+t} \arrow[r, "\Omega^t i_{k+t}"] & \Omega^t\Sigma^{k+t}\mathbb{H}P^2 \arrow[r, "\Omega^t p_{k+t}"] & \Omega^t S^{8+k+t}
\end{tikzcd}$$

and it induces the commutative diagram with exact rows,

$$\begin{tikzcd}
\pi_{m+1}(S^{8+k}) \arrow[r, "\partial_{k*m+1}"] \arrow[d, "\theta_{k+t*}"] & \pi_m(F_k) \arrow[r, "i_{k*}"] \arrow[d, "\varphi_{k+t*}"] & \pi_m(\Sigma^k\mathbb{H}P^2) \arrow[r, "p_{k*}"] \arrow[d, "w_{k+t*}"] & \pi_m(S^{8+k}) \arrow[r, "\partial_{k*m}"] \arrow[d, "\theta_{k+t*}"] & \pi_{m-1}(F_k) \arrow[d, "\varphi_{k+t*}"] \\
\pi_{m+1+t}(S^{8+k+t}) \arrow[r] & \pi_{m+t}(F_{k+t}) \arrow[r, "i_{k+t*}"] & \pi_{m+t}(\Sigma^{k+t}\mathbb{H}P^2) \arrow[r, "p_{k+t*}"] & \pi_{m+t}(S^{8+k+t}) \arrow[r] & \pi_{m-1+t}(F_{k+t})
\end{tikzcd}$$

then it induces the commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & cok(\partial_{k_{*_{m+1}}}) \arrow[r, "\bar{i}_{k*}"] \arrow[d, "\overline{\varphi}_{k+t_*}"] & \pi_m(\Sigma^k \mathbb{H}P^2) \arrow[r, "\overline{p}_{k*}"] \arrow[d, "\overline{w}_{k+t_*}"] & Ker(\partial_{k_{*m}}) \arrow[r] \arrow[d, "\overline{\theta}_{k+t_*}"] & 0 \\
0 \arrow[r] & cok(\partial_{k_{*_{m+1+t}}}) \arrow[r, "\bar{i}_{k+t_*}"] & \pi_{m+t}(\Sigma^{k+t} \mathbb{H}P^2) \arrow[r, "\overline{p}_{k+t_*}"] & Ker(\partial_{k_{*_{m+t}}}) \arrow[r] & 0
\end{tikzcd}$$

the exact sequence

$$0 \to cok(\partial_{k_{*_{m+1}}}) \longrightarrow \pi_m(\Sigma^k \mathbb{H}P^2) \longrightarrow Ker(\partial_{k_{*m}}) \to 0$$

is our main tool to compute $\pi_m(\Sigma^k \mathbb{H}P^2)$.

And j_k, j'_k denote the inclusions associated with the homotopy-commutative diagram,

$$\begin{tikzcd}
S^{4+k} \arrow[d, "j_k"'] \arrow[dr, "j'_k"] & \\
F_k \arrow[r, "i_k"'] & \Sigma^k \mathbb{H}P^2
\end{tikzcd}.$$

We set $sk_{3k+17}(F_k) = S^{4+k} \cup_{f_k} e^{2k+11}$, and $\widehat{j}_k$, $\widetilde{j}_k$ always denote the inclusions of the commutative diagram,

$$\begin{tikzcd}
S^{4+k} \arrow[r, "\widetilde{j}_k"] \arrow[dr, "j_k"'] & sk_{3k+17}(F_k) \arrow[d, "\widehat{j}_k"] \\
& F_k
\end{tikzcd}$$

Now, we generalize theorem 2.6.5, in similar way,we have (recall that we regard the 0-fold smash product as S^0, and M_Y denotes the mapping cylinder of $f : X \to Y$),

Theorem 2.6.7 Suppose

$$X \xrightarrow{f} Y \to C_f \xrightarrow{p} \Sigma X \to \cdots$$

is a cofibre sequence of CW complexes, then there exist a fibre sequence of CW complexes

$$J(M_Y, X) \longrightarrow C_f \xrightarrow{p} \Sigma X,$$

and $J(M_Y, X)$ has the same homology type as $\bigvee_{n\geq 0}(Y \wedge X^{\wedge n})$.

Remark 2.6.8 By Theorem 2.6.7, we can get the following immediately .

For a CW complex $Z = S^{m'} \cup_f e^{m+1}$ where $2 \leq m' < m+1$, $J(M_{S^{m'}}, S^m)$ has a CW structure of type $S^{m'} \cup e^{m'+m} \cup e^{m'+2m} \cup e^{m'+3m} \cup \cdots$.

Lemma 2.7.9 Let $Z = Y \cup_f e^{m+1}$ be a CW complex where $Y = sk_m(Z)$ is simply connected. Then for the fibre sequence,

$$\Omega S^{m+1} \xrightarrow{\partial} J(M_Y, S^m) \longrightarrow Z \longrightarrow S^{m+1}$$

there exists homotopy-commutative diagram,

$$\begin{tikzcd}
S^m \arrow[r, "f"] \arrow[d, "\Omega\Sigma"'] & Y \arrow[d, "\subseteq"] \\
\Omega S^{m+1} \arrow[r, "\partial"] & J(M_Y, S^m).
\end{tikzcd}$$

Proof

We have the following homotopy-commutative diagram,

$$\begin{tikzcd}
 & S^m \arrow[rr, "\Omega\Sigma"] & & \Omega S^{m+1} \arrow[dl, "\partial"] \\
 & S^m \arrow[u, "id"] \arrow[r, "\subseteq"] \arrow[dl, "f"'] \arrow[d, "\subseteq"] & J(S^m) \arrow[ur, "\simeq"'] \arrow[d, "\subseteq"'] & \\
Y \arrow[r, "\subseteq", "\simeq"'] & M_Y \arrow[r, "\subseteq"] & J(M_Y, S^m) &
\end{tikzcd}$$

Thus our lemma is established .

Lemma 2.6.10 Let $Z = Y \cup_f e^{m+1}$ be a CW complex where Y is 1-connected and $dim(Y) = m' \leq m$. Then $J_t(M_Y, S^m) = sk_q(J(M_Y, S^m))$ where $q = m(t-1) + m' - 1$.

Proof

In this proof, we write $J_q(M_Y, S^m)$ as J_q for short.

In fact , we only need to notice that $J_q/J_{q-1} \simeq \Sigma^{(q-1)m}Y$ (*see Page* 499 *of* [2]) and $J(M_Y, S^m)) = \cup_q J_q$, our lemma is implied by the following homotopy-commutative diagram immediately, where $\mathcal{D}_q$ are regarded as the pinch maps which pinch J_{q-1} to the basepoint .

$$\begin{tikzcd}
* \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & J_1 \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & J_2 \arrow[r] \arrow[d, "\subseteq"'] & \cdots \\
J_1 \arrow[r, "\subseteq"] \arrow[d, "\mathcal{D}_1"'] & J_2 \arrow[r, "\subseteq"] \arrow[d, "\mathcal{D}_2"'] & J_3 \arrow[r] \arrow[d, "\mathcal{D}_3"'] & \cdots \\
Y \arrow[r] & \Sigma^m Y \arrow[r] & \Sigma^{2m} Y \arrow[r] & \cdots
\end{tikzcd}$$

2.7 A Relation of Cofibre and Fibre Sequences

Theorem 2.7.1 Let $X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{p} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \to \cdots$ be a cofibre sequence of CW complexes, then there is a homotopy-commutative diagram with rows fibre sequences ,

$$\begin{tikzcd} J(M_Y, X) \arrow[r] \arrow[d] & C_f \arrow[r, "p"] \arrow[d] & \Sigma X \arrow[d, "id"'] & \\ \Omega\Sigma Y \arrow[r] & J(C_f, Y) \arrow[r] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y. \end{tikzcd}$$

Proof

We only need to notice that $-\Sigma f \circ id \circ p \simeq *$. Hence the result. □

Remark 2.7.2 From now on, for the following homotopy-commutative diagram constructed by Theorem 1.1, the symbols $\phi_k, \mathcal{B}_k, d_k, \bar{\jmath}_k, \varpi_k$ always denote the maps of the diagram , and $\mathcal{G}_k$ is the fibre of ϖ_k,

$$\begin{tikzcd} F_k \arrow[r] \arrow[d, "\phi_k"'] & \Sigma^k \mathbb{H}P^2 \arrow[r] \arrow[d, "\mathcal{B}_k"'] & S^{8+k} \arrow[d, "id"'] & \\ \Omega S^{5+k} \arrow[r, "d_k"] & \mathcal{G}_k \arrow[r, "\bar{\jmath}_k"] & S^{8+k} \arrow[r, "\varpi_k"] & S^{5+k}. \end{tikzcd}$$

2.8 the Relative James – Hopf Invariant

The k-th (absolute) James-Hopf invariant for a CW complex X is a map $H_k : J(X) \to J(X^{\wedge k})$, and it is a natural transformation from the functor $J(\ \ )$ to the functor $J(\ \ \ ^{\wedge k})$. In [2], professor Brayton Gray generalized it to the relative version.

Theorem 2.8.1 (*Theorem* 7.2 *of* [2]) For a CW complex pair (X, A) where A is a subcomplex of X , there exists Hopf invariants

$$H_n : J(X, A) \to J(X \wedge A^{\wedge(n-1)})$$

which are natural for pairs .

Remark 2.8.2 Since $J(A, A) = J(A)$, for the map $\partial_k : \Omega S^{8+k} \to F_k$, we have the homotopy-commutative diagram where H_2 and H'_2 are the 2nd relative James-Hopf invari-

ants ,

$$\begin{tikzcd}
\Omega S^{8+k} \arrow[r, "\partial_k"] \arrow[d, "\simeq"'] & F_k \arrow[d, "\simeq"] \\
J(S^{8+k}, S^{8+k}) \arrow[r, "\subseteq"] \arrow[d, "H_2'"'] & J(M_{S^{4+k}}, S^{7+k}) \arrow[d, "H_2"] \\
J(S^{16+2k}) \arrow[r] & J(S^{11+2k})
\end{tikzcd}$$

then we have the homotopy-commutative diagram up to several usual identifications .

$$\begin{tikzcd}
\Omega S^{8+k} \arrow[r, "\partial_k"] \arrow[d, "H_2'"] & F_k \arrow[d, "H_2"] \\
\Omega S^{17+2k} \arrow[r] & \Omega S^{12+2k}
\end{tikzcd}$$

2.9 the (Generalized) Whitehead Product

In 1960s, M. Arkowitz established the generalized Whitehead product in [7]

$$[-,-] : [\Sigma A, X] \times [\Sigma B, X] \longrightarrow [\Sigma A \wedge B, X].$$

Now we restate the definition .

Definition 2.9.1[7] Suppose A, B and X are CW complexes, $p_A : A \times B \to A$, $p_B : A \times B \to B$ are the projections. For $[f] \in [\Sigma A, X]$ and $[g] \in [\Sigma B, X]$, let $f' = f\Sigma p_A$, $g' = g\Sigma p_B$, the Whitehead product $[f, g]$ is the homotopy class of the map w satisfying the following homotopy-commutative diagram, where the lower square is strictly commutative and $\alpha \circ j = *$ strictly, and the addition $+$ (maybe not commutative) is the natural product induced by the co-H structure of the suspensions , $[-1]$ are the selfmaps of degree -1,

$$\begin{tikzcd}
 & \Sigma(A \times B) \arrow[rr, "f'\circ[-1]+g'\circ[-1]+f'+g'"] \arrow[d, "id"] & & X \arrow[d, "id"] \\
\Sigma(A \vee B) \arrow[ur, "\subseteq"] \arrow[r, "j", "\subseteq"'] \arrow[dr, "\subseteq"'] & \Sigma(A \times B) \arrow[rr, "\alpha"] \arrow[d, "id"] & & X \arrow[d, "id"] \\
 & \Sigma(A \times B) \arrow[r, "pinch", "\Sigma(A\vee B)"'] & \Sigma(A \wedge B) \arrow[r, "w"] & X
\end{tikzcd}$$

Proposition 2.9.2[13] Let $\alpha \in \pi_{p+1}(X), \beta \in \pi_{q+1}(X)$, $\gamma \in \pi_m(S^p)$, $\delta \in \pi_n(S^q)$ and $\iota_1 = [id] \in \pi_1(S^1)$, then $[\alpha \circ \Sigma\gamma, \beta \circ \Sigma\delta] = [\alpha, \beta] \circ (\iota_1 \wedge \gamma \wedge \delta) = [\alpha, \beta] \circ \Sigma(\gamma \wedge \delta)$.

Lemma 2.9.3 Suppose $k \geq 1$, then , after localization at 2,

$$sk_{18+3k}(F_k) = S^{4+k} \cup_{f_k} e^{11+2k},$$

where $f_k = [\iota_{4+k}, \nu_{4+k}]$ is the Whitehead product,

and after localization at 3,

$$sk_{18+3k}(F_k) = S^{4+k} \cup_{g_k} e^{11+2k},$$

where $g_k = [\iota_{4+k}, \alpha_1(4+k)]$ is the Whitehead product and $\alpha_1(4+k)$ is the generator of $\pi_{7+k}(S^{4+k}) \approx \mathbb{Z}/3$.

Proof

By lemma 2.6.10 , $J_2(M_{S^{4+k}}, S^{7+k}) = sk_{18+3k}(J(M_{S^{4+k}}, S^{7+k})) = sk_{18+3k}(F_k)$ (up to homotopy). Let $i : S^{7+k} \hookrightarrow M_{S^{4+k}}$ be the inclusion. By proposition 2.6-4, $J_2(M_{S^{4+k}}, S^{7+k}) = M_{S^{4+k}} \cup_\gamma C(M_{S^{4+k}} \wedge S^{6+k})$, $\gamma = [1_{M_{S^{4+k}}}, i]$.

After localization at 2, we only need to show that the right square of the following diagram is homotopy-commutative,

$$\begin{tikzcd}
\Sigma(S^{3+k} \wedge S^{6+k}) \arrow[r, "\subseteq"] \arrow[d, "\subseteq"] & \Sigma(S^{3+k} \times S^{6+k}) \arrow[r, "pinch"] \arrow[d, "\subseteq"] & S^{4+k} \wedge S^{7+k} \arrow[r, "{[\iota_{4+k}, \nu_{4+k}]}"] \arrow[d, "\subseteq"] & S^{4+k} \arrow[d, "\subseteq"] \\
\Sigma(M_{S^{3+k}} \wedge S^{6+k}) \arrow[r, "\subseteq"] & \Sigma(M_{S^{3+k}} \times S^{6+k}) \arrow[r, "pinch"] & M_{S^{4+k}} \wedge S^{7+k} \arrow[r, "{[1_{M_{S^{4+k}}}, i]}"] & M_{S^{4+k}}
\end{tikzcd}$$

While we have the following homotopy-commutative diagram where $p_A : S^{3+k} \times S^{6+k} \to S^{3+k}$, $p_B : S^{3+k} \times S^{6+k} \to S^{6+k}$, $p'_A : M_{S^{3+k}} \times S^{6+k} \to M_{S^{3+k}}$ and $p'_B : M_{S^{3+k}} \times S^{6+k} \to S^{6+k}$ are the projections,

$$\begin{tikzcd}
\Sigma(S^{3+k} \times S^{6+k}) \arrow[r, "\Sigma p_A \circ [-1] + \nu_{4+k} \Sigma p_B \circ [-1] + \Sigma p_A + \nu_{4+k} \Sigma p_B"] \arrow[d, "\subseteq"] & S^{4+k} \arrow[d, "\subseteq"] \\
\Sigma(M_{S^{3+k}} \times S^{6+k}) \arrow[r, "\Sigma p'_A \circ [-1] + i \Sigma p'_B \circ [-1] + \Sigma p'_A + i \Sigma p'_B"] & M_{S^{4+k}}
\end{tikzcd}$$

This implies the right square previous diagram is homotopy-commutative. Similarly, after localization at 3, the lemma is also established. $\square$

The following results are given by the second author in [4].

Lemma 2.9.4[4] After localization at 2, $ord([\iota_6, \nu_6]) = 4$, $ord([\iota_8, \nu_8]) = 8$, $ord([\iota_9, \nu_9]) = 2$, $ord([\iota_{10}, \nu_{10}]) = 4$.

Lemma 2.9.5 After localization at 2, up to homotopy,

$$sk_{23}(F_2) = S^6 \cup_{2\bar{\nu}_6} e^{15},\ sk_{29}(F_4) = S^8 \cup_{f_4} e^{19},$$
$$sk_{32}(F_5) = S^9 \cup_{\bar{\nu}_9\nu_{17}} e^{21},\ sk_{35}(F_6) = S^{10} \cup_{\Delta(\nu_{21})} e^{23},$$

where $f_4 = \nu_8\sigma_{11} - 2t'\sigma_8\nu_{15}$, for one odd t'.
And after localization at 3, up to homotopy,

$$sk_{29}(F_4) = S^8 \cup_{g_4} e^{19}$$

where $g_4 = [\iota_8, \alpha_1(8)] \in \pi_{18}(S^8) = \mathbb{Z}/3\{[\iota_8, \alpha_1(8)]\} \oplus \mathbb{Z}/3\{\beta_1(8)\}$.

Proof

After localization at 2, by lemma 2.9.3 and lemma 2.9.4,
$sk_{18+3k}(F_k) = S^{4+k} \cup_{[\iota_{4+k},\nu_{4+k}]} e^{11+2k}$ and the order of $[\iota_{4+k}, \nu_{4+k}]$ we need is known. Since

$\mathrm{Ker}(\Sigma : \pi_{14}(S^6) \to \pi_{15}(S^7)) = \mathbb{Z}/4\{2\bar{\nu}_6\}$, (by page 61 of [1])

$\mathrm{Ker}(\Sigma : \pi_{18}(S^8) \to \pi_{19}(S^9)) = \mathbb{Z}/8\{\nu_8\sigma_{11} - 2t'\sigma_8\nu_{15}\}$ for one odd t', (by page 75 of [1], $\nu_9\sigma_{12} = 2t'\sigma_9\nu_{15}$ for one odd t')

$\mathrm{Ker}(\Sigma : \pi_{20}(S^9) \to \pi_{21}(S^{10})) = \mathbb{Z}/2\{\bar{\nu}_9\nu_{17}\}$, (by page 73 of [1], $\bar{\nu}_{10}\nu_{18} = 0$)

$\mathrm{Ker}(\Sigma : \pi_{23}(S^{10}) \to \pi_{24}(S^{11})) = \mathbb{Z}/4\{\Delta(\nu_{21})\}$, (by page 74 of [1]).

And after localization at 3, according to *formula* (13.1) *of Page* 172 *of* [1], we have $\pi_{18}(S^8) = \mathbb{Z}/3\{[\iota_8, \iota_8]\circ\alpha_1(15)\}\oplus\mathbb{Z}/3\{\beta_1(8)\}$, by Proposition 2.9.2, we have $[\iota_8, \iota_8]\circ\alpha_1(15) = [\iota_8, \iota_8] \circ (\iota_1 \wedge \iota_7 \wedge \alpha_1(7)) = [\iota_8, \alpha_1(8)]$. Then by lemma 2.9.3, our lemma is established immediately.
(In fact, localized at 2, there's the well-known formula $[\iota_n, \alpha_n] = \pm\Delta(\alpha_{2n+1})$ if $\alpha_{n-1} \in \pi_*(S^{n-1})$ and $\alpha_{n-1+k} = \Sigma^k\alpha_{n-1},\ k \in \mathbb{Z}_+$)

2.10 the Secondary Composition

Definition 2.10.1[1] Let $n \geq 0$ be an integer. Consider elements $\alpha \in [\Sigma^n Y, Z], \beta \in [X, Y]$ and $\gamma \in [W, X]$ which satisfy $\alpha \circ \Sigma^n\beta = 0$ and $\beta \circ \gamma = 0$. And let $a : \Sigma^n Y \to Z$, $b : X \to Y$ and $c : W \to X$ be the representations of α,β and γ respectively .Then there exist homotopies $A_t : \Sigma^n X \to Z$ and $B_t : W \to Y$, $0 \leq t \leq 1$, such that $A_0 = a \circ \Sigma^n b$, $B_0 = b \circ c$, $A_1(\Sigma^n X) = *_Z$ and $B_1(W) = *_Y$. Construct a map $H : \Sigma^{n+1}W \to Z$ by formula

$$H(d(w,t)) = \begin{cases} a(\Sigma^n B_{2t-1}(w)), & 0 \leq t \leq \frac{1}{2}, \\ A_{1-2t}(\Sigma^n c(w)), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

where $w \in \Sigma^n W$ and $d : \Sigma^n W \times I \to \Sigma^{n+1} W$ is an identification defining $\Sigma^{n+1}W = \Sigma(\Sigma^n W)$.

Define the secondary composition $\{\alpha, \Sigma^n\beta, \Sigma^n\gamma\}_n \subseteq [\Sigma^{n+1}W, Z]$ as the set of all homotopy classes of the map H given as above.

Remark 2.10.2 The secondary composition $\{\alpha,\beta,\gamma\}_0$ is usually written as $\{\alpha,\beta,\gamma\}$ for short. And if $\{\alpha,\beta,\gamma\} = \{b\}$, it is usually written as $\{\alpha,\beta,\gamma\} = b$ for short.

Lemma 2.10.3[1] $\{\alpha,\beta,\gamma\}_n$ is a coset of the subgroup

$$[\Sigma^n X, Z] \circ \Sigma^{n+1}\gamma + \alpha \circ \Sigma^n[\Sigma W, Y] \leq [\Sigma^{n+1}W, Z]$$

if $[\Sigma^{n+1}W, Z]$ is an abelian group.

Remark 2.10.3 By *Lemma* 2.10.3, if $[\Sigma W, Z]$ is an abelian, the notations

$$\{\alpha,\beta,\gamma\},\ \text{mod}\ A$$

$$\text{and}\ \{\alpha,\beta,\gamma\},\ \text{mod}\ a_\lambda$$

always denote $\{\alpha,\beta,\gamma\}$ is a coset of the subgroup A (generated by $a_\lambda, \lambda \in \Lambda$) .

Lemma 2.10.4[1] $\{\alpha, \Sigma^n\beta, \Sigma^n\gamma\}_n \subseteq \{\alpha, \Sigma^m(\Sigma^{n-m}\beta), \Sigma^m(\Sigma^{n-m}\gamma)\}_m$ if $n \geq m \geq 0$. And $-\Sigma(\{\alpha, \Sigma^n\beta, \Sigma^n\gamma\}_n) \subseteq \{\Sigma\alpha, \Sigma^{n+1}\beta, \Sigma^{n+1}\gamma\}_{n+1}$ if $n \geq 0$.

Proposition 2.10.5

(i)[1] For the sequence where $\alpha\beta = \beta\gamma = \gamma\delta = 0$,

$$U \xrightarrow{\delta} W \xrightarrow{\gamma} Z \xrightarrow{\beta} Y \xrightarrow{\alpha} X$$

there's equation $\alpha \circ \{\beta,\gamma,\delta\} = -\{\alpha,\beta,\gamma\} \circ \Sigma\delta.$

(ii) (*By Proposition* 1.7 *of Page* 13 , [1] , *Toda*)

For any cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow[inclu.]{i} C_f \xrightarrow[pinch]{p} \Sigma X \longrightarrow \cdots,$$

there always $[id_{\Sigma X}] \in \{p, i, f\}$

(iii) $\Sigma^2\sigma' = 2\sigma_9,\ \zeta_5 \in \{\nu_5, 8\iota_8, \Sigma\sigma'\}$, (*Page* 59 *of* [1]);

$\zeta_6 \in \pm\{\nu_6, 2\sigma_9, 8\iota_{16}\}$ (immediately got by $\Sigma^\infty|_{\langle\zeta_5\rangle}$ is an isomorphism)

$$\pm[\iota_6,\iota_6]=\Delta(\iota_{13})\in\{\nu_6,\eta_9,2\iota_{10}\}\ \ (Lemma\ 5.10\ of\ Page\ 45\ of\ \ [1])$$

(iv) $\sigma''' = \{\nu_5, 2\nu_8, 4\iota_8\}$ ((9) $of\ Page$ 85 of [10] , $N.Oda$)

2.11 the Quasi−quarternionic Space

We introduce the quasi-quarternionic space , which builds the connection of the stable homotopy groups of the complex projective spaces and the quarternionic projective spaces.

Proposition 2.11 ([3],[5]) There is a standard fibration $\gamma_n : \mathbb{C}P^{2n-1} \to \mathbb{H}P^{n-1}$ with cofibre Q_n, named the quasi-quarternionic space, and Q_n has a CW structure of type $S^3 \cup e^7 \cup \cdots \cup e^{4n-1}$.

2.12 Functorial Decompositions of Self Smashes of Spaces

Let f be a self map of X, then the hocolimit of $X\xrightarrow{f}X\xrightarrow{f}X\rightarrow\ \cdots$ is the mapping telescope $(\coprod_{i\geq 0} X\times[i,i+1])/(x,i+1)\sim(f(x),i)$. In fact ,this is a special case of a homotopy colimit, and we write hocolim$_f X$ to donate it. The powerful property of homotopy colimits is that they commute with the homology functor.

Now let the symmetric group S_n act on $X^{\wedge n}$ by permuting positions. Thus, for each $\sigma\in S_n$ we have a map σ: $X^{\wedge n}\to X^{\wedge n}$.Then we obtain a map $h:\Sigma X^{\wedge n}\to\Sigma X^{\wedge n}$ for any $h\in\mathbb{Z}_{(p)}[S_n]$ by using the group structure in $[\Sigma X^{\wedge n},\Sigma X^{\wedge n}]$. Let $1=\Sigma_\alpha e_\alpha$ be an orthogonal decomposition of the identity in $\mathbb{Z}_{(p)}[S_n]$ in terms of primitive idempotents. The composition

$$\Sigma X^{\wedge n}\xrightarrow{\mu'}\bigvee_\alpha\Sigma X^{\wedge n}\to\bigvee_\alpha \mathrm{hocolim}_{e_\alpha}\Sigma X^{\wedge n}$$

(μ' is the natural comultiplication of the suspension) is a homotopy equivalence because its induced map on the singular chains over *p-local* integers is a homotopy equivalence (see [8] for details).

2.13 The Snake Lemma

We use *cok* to denote the cokernels.

Lemma 2.13 (the Snake Lemma, *Corollary* 6.12 *of* [14])

Given a commutative diagram of abelian groups (or $\mathbb{Z}_{(p)}$ modules for one prime p) with exact rows, where f, g and h are homomorphisms,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \end{tikzcd}$$

there exists the exact sequence

$$0 \to Ker(f) \to Ker(g) \to Ker(h) \to cok(f) \to cok(g) \to cok(h) \to 0.$$

3. The Homotopy Groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ after Localization at 3

In this chapter ,without special accent, all cases are after localization at 3 whenever we say after localization at 3 or not.

Firstly we list some results about $\pi_i^s(\mathbb{H}P^\infty)$ due to Aruns Liulevicius.

Proposition 3.1[6]

$$\pi^s_{2i-1}(\mathbb{H}P^\infty)=0, \quad 2i-1 \leq 13$$
$$\pi^s_8(\mathbb{H}P^\infty) \approx \pi^s_{12}(\mathbb{H}P^\infty) \approx \mathbb{Z}_{(3)}$$
$$\pi^s_{15}(\mathbb{H}P^\infty) \approx \mathbb{Z}/3.$$

Lemma 3.2 $\quad \pi^s_{11}(\mathbb{H}P^2) \approx \mathbb{Z}/9$, $\pi^s_{10}(\mathbb{H}P^2)=0$

Proof.

The sequence of spaces $S^4 \hookrightarrow \mathbb{H}P^2 \to S^8$ induces a exact sequence of stable homotopy groups,

$$\begin{array}{ccccccccccc} \cdots \to & \pi^s_{12}(S^8) & \to & \pi^s_{11}(S^4) & \to & \pi^s_{11}(\mathbb{H}P^2) & \to & \pi^s_{11}(S^8) & \to & \pi^s_{10}(S^4) & \to \cdots \\ & \parallel & & \parallel & & & & \parallel & & \parallel & \\ & 0 & & \mathbb{Z}/3 & & & & \mathbb{Z}/3 & & 0 & \end{array}$$

Hence $|\pi^s_{11}(\mathbb{H}P^2)|=9$, this gives $\pi^s_{11}(\mathbb{H}P^2)=\mathbb{Z}/9$ or $\mathbb{Z}/3 \oplus \mathbb{Z}/3$.

It is known that $\pi^s_{11}(\mathbb{H}P^\infty)=0$. Now let $\mathbb{H}P^\infty_3 = \mathbb{H}P^\infty/\mathbb{H}P^2$, the sequence of spaces $\mathbb{H}P^2 \hookrightarrow \mathbb{H}P^\infty \to \mathbb{H}P^\infty_3$ induces an exact sequence of the stable homotopy groups,

$$\begin{array}{ccccc} \pi^s_{12}(\mathbb{H}P^\infty_3) & \to & \pi^s_{11}(\mathbb{H}P^2) & \to & \pi^s_{11}(\mathbb{H}P^\infty) \\ \| & & & & \| \\ \mathbb{Z}_{(3)} & & & & 0 \end{array}$$

So there is a surjective homomorphism from $\mathbb{Z}_{(3)}$ to $\pi^s_{11}(\mathbb{H}P^2)$. Then $\pi^s_{11}(\mathbb{H}P^2) = \mathbb{Z}/9$. And $\pi^s_{10}(\mathbb{H}P^2)$=0 is got by the following exact sequence immediately.

$$\begin{array}{ccccccc} \pi^s_{10}(S^4) & \to & \pi^s_{10}(\mathbb{H}P^2) & \to & \pi^s_{10}(S^8) & \to & \pi^s_{10}(S^4) \\ \| & & & & \| & & \| \\ 0 & & & & 0 & & 0 \end{array}$$

□

Lemma 3.3 Let $\mathbb{H}P^3 = \mathbb{H}P^2 \cup_h e^{12}$ with the attaching map $h : S^{11} \to \mathbb{H}P^2$ and $k \geq 0$, then for any $[f] \in \pi_{11+k}(\Sigma^k\mathbb{H}P^2)$, $[\Sigma^k h] \neq 3[f]$.

Proof

To obtain a contradiction, suppose $\exists k \geq 0$, and $\exists [f] \in \pi_{11+k}(\Sigma^k\mathbb{H}P^2)$, such that $[\Sigma^k h] = 3[f]$. Let $q : \Sigma^k\mathbb{H}P^2 \to \Sigma^k\mathbb{H}P^2/\Sigma^k\mathbb{H}P^1$ be the pinch map, we observe the space

$$\Sigma^k\mathbb{H}P^3/\Sigma^k\mathbb{H}P^1 = S^{8+k} \cup_{q\circ\Sigma^k h} e^{12+k}.$$

According to the assumption and $\pi_{11+k}(S^{8+k}) \approx \mathbb{Z}/3$, we have

$$[q \circ \Sigma^k h] = [q] \circ 3[f] = 3([q] \circ [f])=0 .$$

This gives $\Sigma^k\mathbb{H}P^3/\Sigma^k\mathbb{H}P^1 \simeq S^{8+k} \vee S^{12+k}$. However, $\widetilde{H}_*(\Sigma^k\mathbb{H}P^3/\Sigma^k\mathbb{H}P^1)$ $=\mathbb{Z}/3\{a,\ b\}$ ($|a| = 12 + k$, $|b| = 8 + k$), the Steenord operation is given by $P^1_*(a) = b$ (this could be seen in *Example* 4.L.4 *of Page* 492 *of* [12]), while $\widetilde{H}_*(S^{8+k} \vee S^{12+k})$ only has trivial Steenord operation. Thus $\Sigma^k\mathbb{H}P^3/\Sigma^k\mathbb{H}P^1$ and $S^{8+k} \vee S^{12+k}$ are not homotopy equivalent. This forces $[\Sigma^k h] \neq 3[f]$. □

Lemma 3.4 Before localization , suppose $j_{k_*} : \pi_i(S^{4+k}) \to \pi_i(F_k)$, $j_{k+t_*} : \pi_{i+t}(S^{4+k+t}) \to \pi_{i+t}(F_{k+t})$ and $\Sigma^t : \pi_i(S^{4+k}) \to \pi_{i+t}(S^{4+k+t})$ are three isomorphisms, then $\varphi_{k+t_*} : \pi_i(F_k) \to \pi_{i+t}(F_{k+t})$ is an isomorphism.

Proof.

In fact, both $\Omega^t j_{k+t} \circ \Omega^t\Sigma^t$ and $\varphi_{k+t} \circ j_k$ induce isomorphisms of homology groups of

dimension $4+k$, hence they are either the inclusion $i : S^{4+k} \hookrightarrow \Omega^t F_{k+t}$ or $-i$ up to homotopy. Thus we have the homotopy-commutative diagram (replace $\Omega^t j_{k+t}$ with $-\Omega^t j_{k+t}$ if need, and still written as $\Omega^t j_{k+t}$) ,

$$\begin{tikzcd} S^{4+k} \arrow[r, "j_k"] \arrow[d, "\Omega^t\Sigma^t"'] & F_k \arrow[d, "\varphi_{k+t}"] \\ \Omega^t S^{4+k+t} \arrow[r, "\Omega^t j_{k+t}"] & \Omega^t F_{k+t} \end{tikzcd}$$

It induces the commutative diagram,

$$\begin{tikzcd} \pi_i(S^{4+k}) \arrow[r, "j_{k*}", "\approx"'] \arrow[d, "\Sigma^t"', "\approx"] & \pi_i(F_k) \arrow[d, "\varphi_{k+t*}"] \\ \pi_{i+t}(S^{4+k+t}) \arrow[r, "j_{k+t*}", "\approx"'] & \pi_{i+t}(F_{k+t}) \end{tikzcd}$$

Hence the lemma is established. □

Lemma 3.5 After localization at 3, $\pi_{15}(F_2) \approx \mathbb{Z}_{(3)}$, $\pi_{16}(F_2) = \mathbb{Z}/9\{j_2\beta_1(6)\}$, $\pi_{17}(F_2) = \mathbb{Z}/9\{j_2\alpha_1'(6)\}$.

Proof

Since $sk_{23}(F_2) = S^6 \cup_{[\iota_6,\alpha_1(6)]} e^{15}$, for the fibre sequence,

$$J(M_{S^6}, S^{14}) \longrightarrow sk_{23}(F_2) \longrightarrow S^{15}$$

we have $sk_{19}(J(M_{S^6}, S^{14})) = S^6$. We have the exact sequence,

$$0 = \pi_{15}(S^6) \longrightarrow \pi_{15}(F_2) \longrightarrow \pi_{15}(S^{15}) \longrightarrow \pi_{14}(S^6) \approx \mathbb{Z}/3$$

This gives $\pi_{15}(F_2) \approx \mathbb{Z}_{(3)}$.

We also have the exact sequence,

$$0 = \pi_{17}(S^{15}) \longrightarrow \pi_{16}(S^6) \longrightarrow \pi_{16}(F_2) \longrightarrow \pi_{16}(S^{15}) = 0$$

so, $\pi_{16}(F_2) \approx \pi_{16}(S^6) \approx \mathbb{Z}/9$, what's more, $\pi_{16}(F_2) = \mathbb{Z}/9\{j_2\beta_1(6)\}$. Since $sk_{23}(F_2) = S^7 \vee S^{16}$, we have the commutative diagram with exact rows,

$$\begin{tikzcd} \pi_{18}(S^{15}) \arrow[r] \arrow[d, "\approx"] & \pi_{17}(S^6) \arrow[r] \arrow[d, "\Sigma^\infty"] & \pi_{17}(F_2) \arrow[r] \arrow[d] & \pi_{17}(S^{15}) = 0 \arrow[d] \\ \pi^s_{18}(S^{15}) \arrow[r] & \pi^s_{17}(S^6) \arrow[r] & \pi^s_{18}(S^7) \oplus \pi^s_{18}(S^{16}) \arrow[r] & \pi^s_{17}(S^{15}) = 0 \end{tikzcd}$$

By Serre's isomorphisms in *Page* 180 *of* [1] and *Theorem* 13.9 *of Page* 180 *of* [1], we have $\Sigma^\infty : \pi_{17}(S^6) \to \pi^s_{17}(S^6)$ is an isomorphism. Hence, $\pi_{17}(F_2) \approx \pi^s_{18}(S^7) \oplus \pi^s_{18}(S^{16}) \approx \mathbb{Z}/9$, and then it's easy to see that $\pi_{17}(F_2) = \mathbb{Z}/9\{j_2\alpha_1'(6)\}$. □

Theorem 3.6 After localization at 3,

$$\pi_{7+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}0, & k\geq 1,\\ \mathbb{Z}/3, & k=0\end{cases},\quad \pi_{8+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}\mathbb{Z}_{(3)}, & k\geq 2,\\ 0, & k=0,1\end{cases}$$

$$\pi_{9+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}\mathbb{Z}_{(3)}, & k=2,\\ 0, & else\end{cases},\quad \pi_{10+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}0, & k\geq 1,\\ \mathbb{Z}/3, & k=0\end{cases}$$

$$\pi_{11+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}\mathbb{Z}/9, & k\geq 1 \text{ and } k\neq 4,\\ \mathbb{Z}/9\oplus\mathbb{Z}_{(3)}, & k=4\\ \mathbb{Z}/3\oplus\mathbb{Z}_{(3)}, & k=0\end{cases},\quad \pi_{12+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}0, & k\geq 0 \text{ and } k\neq 1,\\ \mathbb{Z}_{(3)}, & k=1\end{cases}$$

$$\pi_{13+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}0, & k\geq 0 \text{ and } k\neq 2,\,6\\ \mathbb{Z}_{(3)}, & k=2 \text{ or } 6\end{cases}$$

Proof.

1. $\pi_{7+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{7+k}(\Sigma^k\mathbb{H}P^m)$ $(1\leq m\leq\infty)$ are in the stable range $\Leftrightarrow k\geq 7-6=1$

For $k\geq 1$, $\pi_{7+k}(\Sigma^k\mathbb{H}P^2)\approx\pi_{7+k}(\Sigma^k\mathbb{H}P^\infty)\approx\pi_7^s(\mathbb{H}P^\infty)$=0.

And for $k=0$, $\pi_7(\mathbb{H}P^2)\approx\pi_7(\mathbb{H}P^\infty)\approx\pi_6(S^3)\approx\mathbb{Z}/3$. □

2. $\pi_{8+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{8+k}(\Sigma^k\mathbb{H}P^m)$ $(1\leq m\leq\infty)$ are in the stable range $\Leftrightarrow k\geq 8-6=2$,

For $k\geq 2$, $\pi_{8+k}(\Sigma^k\mathbb{H}P^2)\approx\pi_{8+k}(\Sigma^k\mathbb{H}P^\infty)\approx\pi_8^s(\mathbb{H}P^\infty)\approx\mathbb{Z}_{(3)}$.

For $k=1$, $sk_{12}(F_1)=S^5$, the sequence $F_1\to\Sigma\mathbb{H}P^2\to S^9$ induces the exact sequence of homotopy groups,

$$0=\pi_9(S^5)\to\pi_9(\Sigma\mathbb{H}P^2)\to\pi_9(S^9)\to\pi_8(S^5)\approx\mathbb{Z}/3$$

then we have $\pi_9(\Sigma\mathbb{H}P^2)\approx\mathbb{Z}_{(3)}$.

As for $k=0$, $\pi_8(\mathbb{H}P^2)\approx\pi_8(\mathbb{H}P^\infty)\approx\pi_7(S^3)$=0. $\square$

3. $\pi_{9+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{9+k}(\Sigma^k\mathbb{H}P^m)$ $(1\le m\le\infty)$ is in the stable range $\Leftrightarrow k\ge 9-6=3$, so

For $k\ge3$, $\pi_{9+k}(\Sigma^k\mathbb{H}P^2)$=$\pi_{9+k}(\Sigma^k\mathbb{H}P^\infty)$ =$\pi_9^s(\mathbb{H}P^\infty)$=0.

For $k=2$, $sk_{14}(F_2)=S^6$, so we have the exact sequence of homotopy groups,

$$0=\pi_{12}(S^{10})\rightarrow\pi_{11}(S^6)\rightarrow\pi_{11}(\Sigma^2\mathbb{H}P^2)\rightarrow\pi_{11}(S^{10}=0)$$

so $\pi_{11}(\Sigma^2\mathbb{H}P^2)\approx\pi_{11}(S^6)\approx\mathbb{Z}_{(3)}$.

For $k=1$, $sk_{12}(F_1)=S^5$,we have the exact sequence of homotopy groups,

$$0=\pi_{11}(S^9)\rightarrow\pi_{10}(S^5)\rightarrow\pi_{10}(\Sigma\mathbb{H}P^2)\rightarrow\pi_{10}(S^9)=0$$

so , $\pi_{10}(\Sigma\mathbb{H}P^2)\approx\pi_{10}(S^5)=0$.

For $k=0$, $\pi_9(\mathbb{H}P^2)\approx\pi_9(\mathbb{H}P^\infty)\approx\pi_8(S^3)$=0. $\square$

4. $\pi_{10+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{10+k}(\Sigma^k\mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k\ge 10-6=4$,

For $k\ge4$, $\pi_{10+k}(\Sigma^k\mathbb{H}P^2)\approx\pi_{10}^s(\mathbb{H}P^2)$=0 by Lemma 3.2.

For k=3, since $sk_{16}(F_3)=S^7$, we have the the exact sequence of homotopy groups,

$$0=\pi_{13}(S^7)\rightarrow\pi_{13}(\Sigma^3\mathbb{H}P^2)\rightarrow\pi_{13}(S^{11})\rightarrow\pi_{12}(S^7)=0$$

Then by exactness,$\pi_{13}(\Sigma^3\mathbb{H}P^2)\approx\pi_{13}(S^{11})=0$.

For $k=2$, since $sk_{14}(F_2)=S^7$ we have the exact sequence of homotopy groups,

$$0=\pi_{12}(S^7)\rightarrow\pi_{12}(\Sigma^2\mathbb{H}P^2)\rightarrow\pi_{12}(S^{10})=0$$

hence $\pi_{12}(\Sigma^2\mathbb{H}P^2)$=0.

For $k=1$, $sk_{20}(F_1)=S^5\vee S^{13}$ we observe the exact sequence

$$0=\pi_{11}(S^5)\rightarrow\pi_{11}(\Sigma\mathbb{H}P^2)\rightarrow\pi_{11}(S^{10})=0$$

thus $\pi_{11}(\Sigma\mathbb{H}P^2)$=0.

For $k=0$, $\pi_{10}(\mathbb{H}P^2)\approx\pi_{10}(\mathbb{H}P^\infty)\approx\pi_9(S^3)\approx\mathbb{Z}/3$. $\square$

5. $\pi_{11+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{11+k}(\Sigma^k\mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k\ge 11-6=5$.

For $k\ge5$, by Lemma 3.2, $\pi_{11+k}(\Sigma^k\mathbb{H}P^2)\approx\pi_{11}^s(\mathbb{H}P^2)\approx\mathbb{Z}/9$. And according to Lemma 3.3, we have $\pi_{11+k}(\Sigma^k\mathbb{H}P^2)=\mathbb{Z}/9\{[\Sigma^k h]\}$, that is , $[\Sigma^k h]$ is a generator of $\pi_{11+k}(\Sigma^k\mathbb{H}P^2)$

where $k \geq 5$.

For $k = 3$, we observe the exact sequence

$$0{=}\pi_{15}(S^{11}){\to}\pi_{14}(F_3){\to}\pi_{14}(\Sigma^3\mathbb{H}P^2) \to \pi_{14}(S^{11}) \to\pi_{13}(F_3)$$

Since $sk_{16}(F_3) = S^7$, so $\pi_{14}(F_3) = \pi_{14}(S^7) \approx \mathbb{Z}/3$, and $\pi_{13}(F_3) \approx \pi_{13}(S^7) = 0$. Then by exactness we get $|\pi_{14}(\Sigma^3\mathbb{H}P^2)|{=}9$. So $\pi_{14}(\Sigma^3\mathbb{H}P^2) \approx \mathbb{Z}/3 \oplus \mathbb{Z}/3$ *or* $\mathbb{Z}/9$. Now , to obtain a contradiction, suppose

$$\pi_{14}(\Sigma^3\mathbb{H}P^2) \approx \mathbb{Z}/3 \oplus \mathbb{Z}/3.$$

Since any element of $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ is of order 3 or 0. Hence $3[\Sigma^3 h] = 0$. Then $3[\Sigma^5 h] = 0$, while $[\Sigma^5 h] \in \pi_{16}(\Sigma^5\mathbb{H}P^2) \approx \mathbb{Z}/9$ is a generator. This forces $\pi_{14}(\Sigma^3\mathbb{H}P^2) \approx \mathbb{Z}/9$.What's more , by lemma 3.3 ,we know that $\pi_{14}(\Sigma^3\mathbb{H}P^2)$ is generated by $[\Sigma^3 h]$.

For $k = 4$, we observe the exact sequence of homotopy groups,

$$0 = \pi_{16}(S^{12}){\to}\pi_{15}(F_4){\to}\pi_{15}(\Sigma^4\mathbb{H}P^2) \to \pi_{15}(S^{12}) \to\pi_{14}(F_4) = 0$$

Since $sk_{18}(F_4) = S^8$, so $\pi_{15}(F_4) \approx \pi_{15}(S^8) \approx \mathbb{Z}/3 \oplus \mathbb{Z}_{(3)}$, and $\pi_{15}(S^{12}) \approx \mathbb{Z}/3$, then by Lemma 2.2-2, we get

$\pi_{15}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/3,\ \mathbb{Z}_{(3)} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$ *or* $\mathbb{Z}_{(3)} \oplus \mathbb{Z}/9$.

Since $ord([\Sigma^3 h]) = ord([\Sigma^5 h]) = 9$. So, $ord([\Sigma^4 h]) = 9$. It forces $\pi_{15}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}/9 \oplus \mathbb{Z}_{(3)}$. What's more, we have $\pi_{15}(\Sigma^4\mathbb{H}P^2) = \mathbb{Z}/9\{\Sigma^4 h\} \oplus \mathbb{Z}_{(3)}\{j'_4 \circ [\iota_8, \iota_8]\}$.

For $k = 2$, since $sk_{14}(F_2) = S^6$, so $\pi_{13}(F_2) \approx \mathbb{Z}/3$, $\pi_{12}(F_2) = 0$. And we have $\pi_{13}(S^{10}) = \mathbb{Z}/3$. We consider the exact sequence

$$0 = \pi_{14}(S^{10}) \to \pi_{13}(F_2){\to} \pi_{13}(\Sigma^2\mathbb{H}P^2) \to \pi_{13}(S^{10}){\to}\pi_{12}(F_2) = 0$$

Then by exactness we get $|\pi_{13}(\Sigma^2\mathbb{H}P^2)|{=}9$. In the same way as the case of $k = 3$ shown , we have $\pi_{13}(\Sigma^2\mathbb{H}P^2) \approx \mathbb{Z}/9$.

For $k = 1$, since $sk_{20}(F_1) = S^5 \vee S^{13}$, so $\pi_{12}(F_1) \approx \mathbb{Z}/3$, $\pi_{11}(F_1) = 0$. And we have $\pi_{12}(S^9) \approx \mathbb{Z}/3$. We consider the exact sequence

$$0 = \pi_{13}(S^9) \to \pi_{12}(F_1){\to} \pi_{12}(\Sigma\mathbb{H}P^2) \to \pi_{12}(S^9){\to}\pi_{11}(F_1) = 0$$

Then by exactness we get $|\pi_{13}(\Sigma^2\mathbb{H}P^2)|{=}9$. In the same way as the case of $k = 3$ shown, we have $\pi_{12}(\Sigma\mathbb{H}P^2) \approx \mathbb{Z}/9$.

For $k = 0$, $\pi_{11}(\mathbb{H}P^2) \approx \pi_{11}(S^{11}) \oplus \pi_{10}(S^3) \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/3$. $\square$

6. $\pi_{12+k}(\Sigma^k\mathbb{H}P^2)$

For $k \geq 3$ or $k = 1$, by the exact sequence

$$0 = \pi_{13+k}(S^{8+k}) \to \pi_{12+k}(F_k) \to \pi_{12+k}(\Sigma^k \mathbb{H}P^2) \to \pi_{12+k}(S^{8+k}) = 0,$$

we have $\pi_{12+k}(\Sigma^k \mathbb{H}P^2) \approx \pi_{12+k}(F_k)$. Since $sk_{10+2k}(F_k) = S^{4+k}$ and $sk_{20}(F_1) = S^5 \vee S^{13}$, then we get the results .

For $k = 0$, $\pi_{12}(\mathbb{H}P^2) \approx \pi_{12}(S^{11}) \oplus \pi_{11}(S^3) = 0$. $\square$

7. $\pi_{13+k}(\Sigma^k \mathbb{H}P^2)$

For $k \geq 3$ or $k = 1$, by the exact sequence

$$0 = \pi_{14+k}(S^{8+k}) \to \pi_{13+k}(F_k) \to \pi_{13+k}(\Sigma^k \mathbb{H}P^2) \to \pi_{13+k}(S^{8+k}) = 0,$$

we have $\pi_{13+k}(\Sigma^k \mathbb{H}P^2) \approx \pi_{13+k}(F_k)$. We have known that $\pi_{15}(F_2) \approx \mathbb{Z}_{(3)}$, $sk_{10+2k}(F_k) = S^{4+k}$, $sk_{20}(F_1) = S^5 \vee S^{13}$ and $sk_{26}(F_3) = S^7 \vee S^{17}$, then we get the results .

For $k = 0$, $\pi_{13}(\mathbb{H}P^2) \approx \pi_{13}(S^{11}) \oplus \pi_{12}(S^3) = 0$. $\square$

Recall that Q_3 is the cofibre of the standard fibration $\mathbb{C}P^5 \to \mathbb{H}P^2$ introduced in 2.11.

Lemma 3.7 After localization at 3,

$$\pi^s_{15}(\mathbb{C}P^5) \approx \mathbb{Z}/27, \ \pi^s_{15}(S^7 \cup_f e^{11}) = 0 \ \text{(for any } f\text{)} \ , \ \pi^s_{15}(\mathbb{H}P^3_2) \approx \mathbb{Z}/9, \ \ \pi^s_{15}(Q_3) = 0.$$

Proof. $\pi^s_{15}(\mathbb{C}P^5) \approx \mathbb{Z}/27$ has been shown in *proposition 3.5 of* [3] ,*page 202*.

For any map $f : S^{10} \to S^7$, since $\pi_{10}(S^7) \approx \mathbb{Z}/3$, and we observe the Steenord module structure of $\widetilde{H}_*(\Sigma^3 \mathbb{H}P^2)$, we get $S^7 \cup_f e^{11} \simeq \Sigma^3 \mathbb{H}P^2$ or $S^7 \vee S^{11}$. While $\pi^s_{15}(\Sigma^3 \mathbb{H}P^2) = \pi^s_{12}(\mathbb{H}P^2) = 0$, $\pi^s_{15}(S^7 \vee S^{11})$=0, hence $\pi^s_{15}(S^7 \cup_f e^{11}) = 0$.

By observing the of the Steenord module structure of $\widetilde{H}_*(\mathbb{H}P^3_2)$, we have $\mathbb{H}P^3_2 \simeq \Sigma^4 \mathbb{H}P^2$. So $\pi^s_{15}(\mathbb{H}P^3_2) \approx \pi^s_{15}(\Sigma^4 \mathbb{H}P^2) \approx \pi^s_{11}(\mathbb{H}P^2) \approx \mathbb{Z}/9$.

The sequence $S^3 \hookrightarrow Q_3 \to S^7 \cup e^{11}$ induces the exact sequence of stable homotopy groups, $0 = \pi^s_{15}(S^3) \to \pi^s_{15}(Q_3) \to \pi^s_{15}(S^7 \cup e^{11}) = 0$, hence $\pi^s_{15}(Q_3) = 0$. $\square$

Lemma 3.8 After localization at 3, $\pi_{19}(S^8 \cup_f e^{19}) \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/9$, for any f.

Proof.

After localization at 3, let $f : S^{18} \to S^8$ be a map, and $p : S^8 \cup_f e^{19} \to S^{19}$ be the pinch map with homotopy fibre G. According to Remark 2.6.8, we have $sk_{26}(G) = S^8$. Then we observe the exact sequence,

$$0 = \pi_{20}(S^{19}) \to \pi_{19}(S^8) \to \pi_{19}(S^8 \cup_f e^{19}) \to \pi_{19}(S^{19}) \to \pi_{18}(G) \approx \mathbb{Z}/3,$$

since $\pi_{19}(S^8) \approx \mathbb{Z}/9$, we get $\pi_{19}(S^8 \cup_f e^{19}) \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/9$. □

Theorem 3.9 After localization at 3,

$$\pi_{14+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/3, & k \geq 5 \ or \ k = 4,\ 2 \\ \mathbb{Z}_{(3)} \oplus \mathbb{Z}/3, & k = 3 \\ \mathbb{Z}/9, & k = 1 \\ \mathbb{Z}/3 \oplus \mathbb{Z}/3, & k = 0 \end{cases}$$

$$\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/27, & k \geq 9 \ or \ k = 7, 6, 5 \ or \ 3 \\ \mathbb{Z}/27 \oplus \mathbb{Z}_{(3)}, & k = 4 \ or \ 8 \\ \mathbb{Z}/9, & k = 2 \\ \mathbb{Z}/9 \oplus \mathbb{Z}/3, & k = 1 \\ \mathbb{Z}/3, & k = 0 \end{cases}$$

Proof. We first give the calculation of $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$, since our calculation of $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$ needs some results of it. (However, in fact, it's also feasible if we calculate $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$ firstly in other ways).

1. $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$.

$\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k \geq 15 - 6 = 9$.

For $k \geq 9$, we observe the diagram with homotopy-commute squares ,

$$\begin{tikzcd} S^4 \arrow[r] \arrow[d] & \mathbb{H}P^2 \arrow[r] \arrow[d] & S^8 \arrow[d] \\ S^4 \arrow[r] & \mathbb{H}P^3 \arrow[r] & \mathbb{H}P^3_2 \end{tikzcd} ,$$

it induces the commutative diagram with exact rows ,

$$\begin{tikzcd}
0=\pi^s_{16}(S^8) \arrow[r] \arrow[d] & \pi^s_{15}(S^4) \arrow[r, "i"] \arrow[d, "id"] & \pi^s_{15}(\mathbb{H}P^2) \arrow[r, "p"] \arrow[d] & \pi^s_{15}(S^8) \arrow[r, "q"] \arrow[d, "\ell"] & \pi^s_{14}(S^4) \arrow[d, "id"] \\
\pi^s_{16}(\mathbb{H}P^3_2) \arrow[r] & \pi^s_{15}(S^4) \arrow[r] & \pi^s_{15}(\mathbb{H}P^3) \arrow[r] & \pi^s_{15}(\mathbb{H}P^3_2) \arrow[r, "\gamma"] & \pi^s_{14}(S^4).
\end{tikzcd}$$

Since the composition $\gamma \circ \ell$:

$$\begin{tikzcd}
\pi^s_{15}(S^8) \arrow[r, "\ell"] \arrow[d, equal] & \pi^s_{15}(\mathbb{H}P^3_2) \arrow[r, "\gamma"] \arrow[d, equal] & \pi^s_{14}(S^4) \arrow[d, equal] \\
\mathbb{Z}/3 & \mathbb{Z}/9 & \mathbb{Z}/3
\end{tikzcd}$$

is trivial, we have q=0 , then p is onto ,and obviously i is into .

Because $\pi^s_{15}(S^4) \approx \mathbb{Z}/9$, $\pi^s_{15}(S^8) \approx \mathbb{Z}/3$, we get $\pi^s_{15}(\mathbb{H}P^2) \approx \mathbb{Z}/27$ *or* $\mathbb{Z}/9 \oplus \mathbb{Z}/3$. We consider the exact sequence

$$\mathbb{Z}/27 \approx \pi^s_{15}(\mathbb{C}P^5) \to \pi^s_{15}(\mathbb{H}P^2) \to \pi^s_{15}(\mathbb{H}Q^3) = 0,$$

it gives $\pi^s_{15}(\mathbb{H}P^2)$ is a quotient group of $\mathbb{Z}/27$ up to isomorphism. Hence $\pi^s_{15}(\mathbb{H}P^2) \approx \mathbb{Z}/27$.

For $k = 7$, since $sk_{24}(F_7) = S^{11}$, $sk_{28}(F_9) = S^{13}$, and by observing the diagram of the fibre sequences with homotopy-commutative squares,

$$\begin{tikzcd}
F_7 \arrow[r] \arrow[d, "\varphi_{7+2}"] & \Sigma^7\mathbb{H}P^2 \arrow[r] \arrow[d, "\omega_{7+2}"] & S^{15} \arrow[d] \\
\Omega^2 F_9 \arrow[r] & \Omega^2\Sigma^9\mathbb{H}P^2 \arrow[r] & \Omega^2 S^{17}
\end{tikzcd}$$

it induces the commutative diagram with exact rows ,

$$\begin{tikzcd}
0=\pi_{23}(S^{15}) \arrow[r] \arrow[d] & \pi_{22}(F_7) \arrow[r] \arrow[d, "\varphi_{7+2_{*22}}"] & \pi_{22}(\Sigma^7\mathbb{H}P^2) \arrow[r] \arrow[d, "\omega_{7+2_*}"] & \pi_{22}(S^{15}) \arrow[r] \arrow[d, "\approx"] & \pi_{21}(F_7) \arrow[d, "\varphi_{7+2_{*21}}"] \\
0=\pi_{25}(S^{17}) \arrow[r] & \pi_{24}(F_9) \arrow[r] & \pi_{24}(\Sigma^9\mathbb{H}P^2) \arrow[r] & \pi_{24}(S^{17}) \arrow[r] & \pi_{23}(F_9)
\end{tikzcd}$$

By lemma 2.4.6 and lemma 3.4, we get that $\varphi_{7+2_{*22}}$ and $\varphi_{7+2_{*21}}$ are isomorphisms. Then, by the Five Lemma, $\omega_{7+2_*} : \pi_{22}(\Sigma^7\mathbb{H}P^2) \to \pi_{24}(\Sigma^9\mathbb{H}P^2)$ is an isomorphism.

For $k = 6, 5$ *or* 3, in the same way as the case of $k = 7$ shown, we could also get $\pi_{15+k}(\Sigma^k\mathbb{H}P^2) \approx \pi_{24}(\Sigma^9\mathbb{H}P^2)$ by observing the diagram of the fibre sequences with homotopy-commute squares and the exact sequences of homotopy groups with commutative squares it induced.

$$\begin{tikzcd}
F_k \arrow[r] \arrow[d] & \Sigma^k \mathbb{H}P^2 \arrow[r] \arrow[d] & S^{8+k} \arrow[d] \\
\Omega^{9-k}F_k \arrow[r] & \Omega^{9-k}\Sigma^9\mathbb{H}P^2 \arrow[r] & \Omega^{9-k}S^{17}
\end{tikzcd}$$

For $k = 8$, it's a little more complex than the above cases. We also observe the diagram of the fibre sequences with homotopy-commute squares,

$$\begin{tikzcd}
F_7 \arrow[r] \arrow[d] & \Sigma^7 \mathbb{H}P^2 \arrow[r] \arrow[d] & S^{15} \arrow[d] \\
\Omega F_8 \arrow[r] & \Omega\Sigma^8\mathbb{H}P^2 \arrow[r] & \Omega S^{16}
\end{tikzcd}$$

it induces two exact sequences of homotopy groups with commutative squares .

$$\begin{tikzcd}
 & \mathbb{Z}/9 \arrow[d, equal] & \mathbb{Z}/27 \arrow[d, equal] & \mathbb{Z}/3 \arrow[d, equal] & \\
0 = \pi_{23}(S^{15}) \arrow[r] \arrow[d] & \pi_{22}(F_7) \arrow[r, "i_{7_*}"] \arrow[d, "\varphi_{7+1_{*22}}"] & \pi_{22}(\Sigma^7\mathbb{H}P^2) \arrow[r, "p_{7_*}"] \arrow[d, "w_{7+1_*}"] & \pi_{22}(S^{15}) \arrow[r, "\partial_{7_*}"] \arrow[d, "\approx"] & \pi_{21}(F_7) \arrow[d, "\varphi_{7+1_{*21}}"] \\
0 = \pi_{24}(S^{16}) \arrow[r] & \pi_{23}(F_8) \arrow[r, "i_{8_*}"] \arrow[d, equal] & \pi_{23}(\Sigma^8\mathbb{H}P^2) \arrow[r, "p_{8_*}"] & \pi_{23}(S^{16}) \arrow[r, "\partial_{8_*}"] \arrow[d, equal] & \pi_{22}(F_8) \\
 & \mathbb{Z}/9 \oplus \mathbb{Z}_{(3)} & & \mathbb{Z}/3 &
\end{tikzcd}$$

By lemma 3.4, we get $\varphi_{7+1_{*21}}$ is an isomorphism. By exactness, $\partial_{7_*} = 0$, hence $\partial_{8_*} = 0$, so p_{8_*} is onto. Then by Lemma 2.2.2, we get $\pi_{23}(\Sigma^8\mathbb{H}P^2) \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/9$, $\mathbb{Z}_{(3)} \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/3$, *or* $\mathbb{Z}_{(3)} \oplus \mathbb{Z}/27$. By the Snake Lemma , w_{7+1_*} is into. So $\pi_{23}(\Sigma^8\mathbb{H}P^2) \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/27$. In addition, we get that $p_{k_*} : \pi_{15+k}(\Sigma^k\mathbb{H}P^2) \to \pi_{15+k}(S^{8+k})$ is onto if $k \geq 5$ or $k = 3$.

For $k = 4$, we know that $sk_{29}(F_4) = S^8 \cup e^{19}$, $\pi_{19}(F_4) = \pi_{19}(S^8 \cup e^{19}) \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/9$. Then in the similar way as the case of $k = 8$ shown, by observing the following diagram of the fibre sequences with homotopy-communicative squares and the exact sequences of homotopy groups with communicative squares it induces , we get $\pi_{19}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/27$.

$$\begin{tikzcd}
F_3 \arrow[r] \arrow[d] & \Sigma^3\mathbb{H}P^2 \arrow[r] \arrow[d] & S^{11} \arrow[d] \\
\Omega F_4 \arrow[r] & \Omega\Sigma^4\mathbb{H}P^2 \arrow[r] & \Omega S^{12}
\end{tikzcd}$$

For $k = 2$, we need $\mathrm{Ker}(\partial_{2_{*17}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/3\{\alpha_2(9)\} \arrow[d, equal] & \mathbb{Z}/9\{\beta_1(6)\} \arrow[d, equal] \\
\pi_{16}(S^9) \arrow[r, "(\alpha_1(6))_*"] \arrow[d, "\approx"'] & \pi_{16}(S^6) \arrow[d, "\approx"] \\
\pi_{17}(S^{10}) \arrow[r, "\partial_{2_{*17}}"] & \pi_{16}(F_2)
\end{tikzcd}$$

By $Page$ 180 of [1], we have $\alpha_1(6) \circ \alpha_2(9) = -3\beta_1(8)$, so $(\alpha_1(8))_*$ is into, successively, $\mathrm{Ker}(\partial_{2_{*17}}) = 0$, then we get the exact sequence,

$$0 = \pi_{18}(S^{10}) \to \pi_{17}(F_2) \to \pi_{17}(\Sigma^2\mathbb{H}P^2) \to 0 \to 0,$$

this gives $\pi_{17}(\Sigma^2\mathbb{H}P^2) \approx \pi_{17}(F_2) \approx \mathbb{Z}/9$.

For $k = 1$, we need $\mathrm{Ker}(\partial_{1_{*16}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/3\{\alpha_2(9)\} \arrow[d, equal] & \mathbb{Z}/9\{j_1\beta_1(6)\} \arrow[d, equal] \\
\pi_{16}(S^9) \arrow[r, "\partial_{1_{*16}}"] \arrow[d, "id"'] & \pi_{15}(S^5 \vee S^{13}) \arrow[d, "\phi_{1*}"] \\
\pi_{16}(S^9) \arrow[r, "(\alpha_1(6))_*"] & \pi_{16}(S^6)
\end{tikzcd}$$

By $formula$ (13.1) of $Page$ 172 of [1], we could get $\Sigma : \pi_{15}(S^5) \to \pi_{16}(S^6)$ is an isomorphism. Since $\alpha_1(6) \circ \alpha_2(9) = -3\beta_1(6)$, so $(\alpha_1(6))_*$ is into, successively, $\mathrm{Ker}(\partial_{1_{*16}}) = 0$, then we get the exact sequence,

$$0 = \pi_{18}(S^{10}) \to \pi_{16}(S^5 \vee S^{13}) \to \pi_{16}(\Sigma\mathbb{H}P^2) \to 0 \to 0,$$

this gives $\pi_{16}(\Sigma\mathbb{H}P^2) \approx \pi_{16}(S^5 \vee S^{13}) \approx \mathbb{Z}/9 \oplus \mathbb{Z}/3$.

For $k = 0$, $\pi_{15}(\mathbb{H}P^2) \approx \pi_{15}(S^{11}) \oplus \pi_{14}(S^3) \approx \mathbb{Z}/3$.

2. $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$

For $k \geq 5$ or $k = 3$, we observe the exact sequence,

$$\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \xrightarrow{p_{k*}} \pi_{15+k}(S^{8+k}) \xrightarrow{\partial_{k*}} \pi_{14+k}(F_k) \xrightarrow{i_{k*}} \pi_{14+k}(\Sigma^k \mathbb{H}P^2) \longrightarrow 0$$

By the computation of $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$, we know $p_{k_*} : \pi_{15+k}(\Sigma^k \mathbb{H}P^2) \to \pi_{15+k}(S^{8+k})$ is onto if $k \geq 5$ or $k = 3$. So we have $\partial_{k*} = 0$ for $k \geq 5$ or $k = 3$, thus

$$i_{k*} : \pi_{14+k}(F_k) \longrightarrow \pi_{14+k}(\Sigma^k \mathbb{H}P^2) \text{ is an isomorphism.}$$

Since $sk_{10+2k}(F_k) = S^{4+k}$, $sk_{20}(F_1) = S^5 \vee S^{13}$ and $sk_{26}(F_3) = S^7 \vee S^{17}$, then we get the group $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$.

For $k = 4$, we need $\mathrm{cok}(\partial_{4_{*19}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/3\{\alpha_2(11)\} \arrow[d, equal] & \mathbb{Z}/3\{\beta_1(8)\} \oplus \mathbb{Z}/3\{[\iota_8, \alpha_1(8)]\} \arrow[d, equal] \\
\pi_{18}(S^{11}) \arrow[r, "(\alpha_1(8))_*"] \arrow[d, "\approx"'] & \pi_{18}(S^8) \arrow[d, "j_{4*}"', "onto"] \\
\pi_{19}(S^{12}) \arrow[r, "\partial_{4_{*19}}"] & \pi_{18}(S^8 \cup e^{19})
\end{tikzcd}$$

By *Page* 180 *of* [1], we have $\alpha_1(8) \circ \alpha_2(11) = -3\beta_1(8) = 0$, so $(\alpha_1(8))_* = 0$, successively, $\mathrm{cok}(\partial_{4_{*19}}) \approx \mathbb{Z}/3$, then we get the exact sequence,

$$0 \longrightarrow \mathbb{Z}/3 \longrightarrow \pi_{18}(\Sigma^4 \mathbb{H}P^2) \longrightarrow \pi_{18}(S^{12}) = 0,$$

this gives $\pi_{18}(\Sigma^4 \mathbb{H}P^2) \approx \mathbb{Z}/3$.

For $k = 2$, we need $\mathrm{cok}(\partial_{2_{*17}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/3\{\alpha_2(9)\} \arrow[d, equal] & \mathbb{Z}/9\{\beta_1(6)\} \arrow[d, equal] \\
\pi_{16}(S^9) \arrow[r, "(\alpha_1(6))_*"] \arrow[d, "\approx"'] & \pi_{16}(S^6) \arrow[d, "\approx"] \\
\pi_{17}(S^{10}) \arrow[r, "\partial_{2_{*17}}"] & \pi_{16}(F_2)
\end{tikzcd}$$

By *Page* 180 *of* [1], we have $\alpha_1(6) \circ \alpha_2(9) = -3\beta_1(6)$, thus $\mathrm{cok}(\partial_{2_{*17}}) \approx \mathbb{Z}/3$, then

we get the exact sequence,

$$0 \longrightarrow \mathbb{Z}/3 \longrightarrow \pi_{16}(\Sigma^2\mathbb{H}P^2) \longrightarrow \pi_{16}(S^{10}) = 0,$$

this gives $\pi_{16}(\Sigma^2\mathbb{H}P^2) \approx \mathbb{Z}/3$.

For $k = 1$, we need $\mathrm{cok}(\partial_{1_{*16}})$. We observe the commutative diagram (by Remak 2.7.2),

$$\begin{tikzcd}
\pi_{16}(S^9) \arrow[r, "\partial_{1_{*16}}"] \arrow[d, "id"'] & \pi_{15}(S^5) \arrow[d, "\Sigma"', "\approx"] \\
\pi_{16}(S^9) \arrow[r, "(\alpha_1(6))_*"] \arrow[d, equal] & \pi_{16}(S^6) \arrow[d, equal] \\
\mathbb{Z}/3\{\alpha_2(9)\} & \mathbb{Z}/9\{\beta_1(6)\}
\end{tikzcd}$$

Here , we use an identification $\pi_{15}(S^5) = \pi_{15}(F_1)$. By *Page* 180 *of* [1], we have $\alpha_1(6) \circ \alpha_2(9) = -3\beta_1(6)$; and $\Sigma : \pi_{15}(S^5) \to \pi_{16}(S^6)$ is an isomorphism by Serre's homomorphism. Thus $\mathrm{cok}(\partial_{1_{*16}}) \approx \mathbb{Z}/3$, then we get the exact sequence,

$$0 \longrightarrow \mathbb{Z}/3 \longrightarrow \pi_{15}(\Sigma\mathbb{H}P^2) \longrightarrow \pi_{15}(S^9) = 0,$$

this gives $\pi_{15}(\Sigma\mathbb{H}P^2) \approx \mathbb{Z}/3$.

For $k = 0$, $\pi_{14}(\mathbb{H}P^2) \approx \pi_{14}(S^{11}) \oplus \pi_{13}(S^3) \approx \mathbb{Z}/3 \oplus \mathbb{Z}/3$. □

4. The Homotopy Groups $\pi_{r+k}(\Sigma^k\mathbb{H}P^2)$ after Localization at 2

In this chapter, without special accent, all cases are after localization at 2 whenever we say after localization at 2 or not.

When we take the Toda Bracket Method, we mainly use the statements in 2.10 of the Preliminaries.

Firstly we list some results due to Aruns Liulevicius and the second author .

Lemma 4.1

(**i**) ([6], *Aruns Liulevicius*)

$$\pi_7^s(\mathbb{H}P^\infty) = 0, \quad \pi_8^s(\mathbb{H}P^\infty) \approx \mathbb{Z}_{(2)}, \pi_9^s(\mathbb{H}P^\infty) \approx \pi_{10}^s(\mathbb{H}P^\infty) \approx \mathbb{Z}/2.$$

(**ii**) ([3], *Juno Mukai*)

Let j' be the inclusion $\Sigma^\infty S^4 \hookrightarrow \Sigma^\infty\mathbb{H}P^2$, then

$\pi^s_{11}(\mathbb{H}P^2) = \mathbb{Z}/4\{\widetilde{(2\nu)}\} \oplus \mathbb{Z}/16\{j' \circ \sigma\}, \widetilde{(2\nu)} \in \langle j', \nu, 2\nu\rangle$
$\pi^s_{13}(\mathbb{H}P^2) \approx (\mathbb{Z}/2)^2, \ \ \pi^s_{14}(\mathbb{H}P^2) \approx \mathbb{Z}/2,$
$\pi^s_{15}(\mathbb{H}P^2) = \mathbb{Z}/128\{\widehat{a}\}, \ \widehat{a} \in \langle j', \nu, \sigma\rangle.$

Equivalently,

$\pi_{16}(\Sigma^5\mathbb{H}P^2) = \mathbb{Z}/4\{\widetilde{(2\nu_{13})}\} \oplus \mathbb{Z}/16\{j'_5 \circ \sigma_9\}, \ \widetilde{(2\nu_{13})} \in \{j'_5, \nu_9, 2\nu_{12}\},$
$\pi_{24}(\Sigma^9\mathbb{H}P^2) = \mathbb{Z}/128\{\widehat{a}_9\}\ , \widehat{a}_9 \in \{j'_9, \nu_{13}, \sigma_{16}\}.$

Remark: In [3], our j' is wrote as i.

By above lemma, immediately we know,

Lemma 4.3 $0 \notin \{j'_9, \nu_{13}, \sigma_{16}\}$, and for any $x \in \{j'_9, \nu_{13}, \sigma_{16}\}$, x is a generator of $\pi_{24}(\Sigma^9\mathbb{H}P^2)$.

Lemma 4.4 $i_{5_*} : \pi_{16}(F_5) \to \pi_{16}(\Sigma^5\mathbb{H}P^2)$ is into given by $j_5 \circ \sigma_9 \mapsto j'_5 \circ \sigma_9$.

Proof

Recall that $sk_{20}(F_5) = S^9$, then there's the commutative diagram

$$\begin{tikzcd} S^9 \arrow[r, "\subseteq", "j'_5"'] \arrow[d, "j_5"', "\subseteq"] & \Sigma^5\mathbb{H}P^2 \arrow[d, "id"] \\ F_5 \arrow[r, "i_5"] & \Sigma^5\mathbb{H}P^2 \end{tikzcd}$$

so we have the commutative diagram of homotopy groups,

$$\begin{tikzcd} \pi_{16}(S^9) \arrow[r, "j'_{5_*}"'] \arrow[d, "j_{5_*}"', "\approx"] & \pi_{16}(\Sigma^5\mathbb{H}P^2) \arrow[d, "id"] \\ \pi_{16}(F_5) \arrow[r, "i_{5_*}"] & \pi_{16}(\Sigma^5\mathbb{H}P^2) \end{tikzcd}$$

Since $\pi_{16}(S^9) = \mathbb{Z}/16\{\sigma_9\}$, $\pi_{16}(F_5) = \mathbb{Z}/16\{j_5 \circ \sigma_9\}$, and by lemma 4.1, $\pi_{16}(\Sigma^5\mathbb{H}P^2) = \mathbb{Z}/4\{a'_9\} \oplus \mathbb{Z}/16\{j'_5 \circ \sigma_9\}$, then we get $i_{5_*} : \pi_{16}(F_5) \to \pi_{16}(\Sigma^5\mathbb{H}P^2)$ is into given by $j_5 \circ \sigma_9 \mapsto j'_5 \circ \sigma_9$. $\square$

Lemma 4.5 $\pi_{17}(F_2) = \mathbb{Z}/8\{j_2\zeta_6\} \oplus \mathbb{Z}/2\{j_2\bar{\nu}_6\nu_{14}\} \oplus \mathbb{Z}/2\{\widehat{j_2}\widehat{\eta^2_{15}}\},$
$\pi_{16}(F_2) = \mathbb{Z}/8\{j_2\nu_6\sigma_9\} \oplus \mathbb{Z}/2\{j_2\eta_6\mu_7\} \oplus \mathbb{Z}/2\{\widehat{j_2}\widehat{\eta_{15}}\},$
$\pi_{15}(F_2) = \mathbb{Z}/2\{j_2\nu^3_6\} \oplus \mathbb{Z}/2\{j_2\mu_6\} \oplus \mathbb{Z}/2\{j_2\eta_6\varepsilon_7\} \oplus \mathbb{Z}_{(2)}\{\breve{a}\}.$
$\pi_{19}(F_4) = \mathbb{Z}/8\{j_4\zeta_8\} \oplus \mathbb{Z}/2\{j_4\bar{\nu}_8\nu_{16}\} \oplus \mathbb{Z}_{(2)}\{\acute{a}\}.$
$\pi_{20}(F_5) = \mathbb{Z}/8\{j_5\zeta_9\}.$

Here, the generators $\acute{a}$, $\widehat{j_2}\widehat{\eta^2_{15}}$ *and* $\widehat{j_2}\widehat{\eta_{15}}$ are stated in the proof of this lemma.

Proof

1. $\pi_{17}(F_2)$

By lemma 2.9.5, we have $sk_{23}(F_2) = J_2(M_{S^6}, S^9) = S^6 \cup_{f_2} e^{15}$ up to homototy, where $f_2 = 2\bar{\nu}_6 \in \pi_{14}(S^6) = \mathbb{Z}/8\{\bar{\nu}_6\} \oplus \mathbb{Z}/2\{\varepsilon_6\}$. We consider the fibre ation sequence,

$$\Omega S^{15} \xrightarrow{\partial'_2} J(M_{S^6}, S^{14}) \xrightarrow{\check{j}_2} S^6 \cup_{f_2} e^{15} \xrightarrow[\check{p}_2]{pinch} S^{15}$$

By Remark 2.6.8, we have $sk_{19}(J(M_{S^6}, S^{14})) = S^6$. Then we have the exact sequence,

$$\pi_{18}(S^{15}) \xrightarrow{\partial'_{2_{*18}}} \pi_{17}(J(M_{S^6}, S^{14})) \longrightarrow \pi_{17}(F_2) \longrightarrow \pi_{17}(S^{15}) \xrightarrow{\partial'_{2_{*17}}} \pi_{16}(J(M_{S^6}, S^{14}))$$

We need $\mathrm{cok}(\partial'_{2_{*18}})$ and $\mathrm{Ker}(\partial'_{2_{*17}})$. Now, we observe the commutative diagram,

$$\begin{tikzcd}
 & \mathbb{Z}/8\{\zeta_6\} \oplus \mathbb{Z}/4\{\bar{\nu}_6\nu_{14}\} \arrow[d, phantom, "\parallel"] \\
\pi_{17}(S^{14}) \arrow[r, "(2\bar{\nu}_6)_*"] \arrow[d, "\approx"'] & \pi_{17}(S^6) \arrow[d, "\approx"] \\
\pi_{18}(S^{15}) \arrow[r, "\partial'_{2_{*18}}"] & \pi_{17}(J(M_{S^6}, S^{14}))
\end{tikzcd}$$

Since $(2\bar{\nu}_6)_*(\nu_{14}) = 2\bar{\nu}_6\nu_{14}$. Thus $\mathrm{cok}(\partial'_{2_{*18}}) = (\mathbb{Z}/8\{j_2\zeta_6\} \oplus \mathbb{Z}/4\{j_2\bar{\nu}_6\nu_{14}\})/ < 2j_2\bar{\nu}_6\nu_{14} >= \mathbb{Z}/8\{j_2\zeta_6\} \oplus \mathbb{Z}/2\{\mathrm{cls}(j_2\bar{\nu}_6\nu_{14})\}$, where $\mathrm{cls}(j_2\bar{\nu}_6\nu_{14}) = j_2\bar{\nu}_6\nu_{14}+ < 2j_2\bar{\nu}_6\nu_{14} >$ and we identify $j_2\zeta_6+ < 2j_2\bar{\nu}_6\nu_{14} >$ with $j_2\zeta_6$.

Then, we observe the commutative diagram,

$$\begin{tikzcd}
\pi_{16}(S^{14}) \arrow[r, "(2\bar{\nu}_6)_*"] \arrow[d, "\approx"'] & \pi_{16}(S^6) \arrow[d, "\approx"] \\
\pi_{17}(S^{15}) \arrow[r, "\partial'_{2_{*17}}"] & \pi_{16}(J(M_{S^6}, S^{14}))
\end{tikzcd}$$

Since $\pi_{16}(S^{14}) \approx \mathbb{Z}/2$, we get $(2\bar{\nu}_6)_* : \pi_{16}(S^{14}) \to \pi_{16}(S^6)$ is zero. Hence, $\mathrm{Ker}(\partial'_{2_{*17}}) = \pi_{17}(S^{15}) = \mathbb{Z}/2\{\eta^2_{15}\}$. So we have the exact sequence,

$$0 \to \mathbb{Z}/8\{j_2\zeta_6\} \oplus \mathbb{Z}/2\{cls(j_2\bar{\nu}_6\nu_{14})\} \longrightarrow \pi_{17}(F_2) \longrightarrow \mathbb{Z}/2\{\eta^2_{15}\} \to 0$$

Since $\Sigma(S^6 \cup_{f_2} e^{15}) = S^7 \vee S^{16}$,(recall that $f_2 \in Ker(\Sigma)$), and then we have the homotopy-commutative diagram with rows fibre sequences,

$$\begin{tikzcd}
J(M_{S^6}, S^{14}) \arrow[r] \arrow[d, "c_2"] & S^6 \cup_{f_5} e^{15} \arrow[r] \arrow[d, "\Omega\Sigma"'] & S^{15} \arrow[d, "\Omega\Sigma"] \\
\Omega J(M_{S^7}, S^{15}) \arrow[r] & \Omega(S^7 \vee S^{16}) \arrow[r] & \Omega S^{16}
\end{tikzcd}$$

By Remark 2.6.8, $sk_{21}(J(M_{S^7}, S^{15})) = S^7$. So we have the commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/8\{\zeta_6\} \oplus \mathbb{Z}/2\{cls(\bar{\nu}_6\nu_{14})\} \arrow[r] \arrow[d, "c_{2_*}"'] & \pi_{17}(S^6 \cup_{f_2} e^{15}) \arrow[r] \arrow[d, "\Sigma"'] & \pi_{17}(S^{15}) \arrow[r] \arrow[d, "\Sigma"', "\approx"] & 0 \\
0 \arrow[r] & \pi_{18}(S^7) \arrow[r] \arrow[d, equal] & \pi_{18}(S^7) \oplus \pi_{18}(S^{16}) \arrow[r] & \pi_{18}(S^{16}) \arrow[r] & 0 \\
 & \mathbb{Z}/8\{\zeta_7\} \oplus \mathbb{Z}/2\{\bar{\nu}_7\nu_{15}\} & & &
\end{tikzcd}$$

Here, c_{2_*} is induced by the suspension homomorphism and it is an isomorphism. By the Five Lemma, $\pi_{17}(S^6 \cup_{f_2} e^{15}) \approx \pi_{18}(S^7) \oplus \pi_{18}(S^{16})$, thus, we get $\pi_{17}(S^6 \cup_{f_2} e^{15}) \approx \mathbb{Z}/8\{\zeta_6\} \oplus \mathbb{Z}/2\{\bar{\nu}_6\nu_{14}\} \oplus \mathbb{Z}/2\{\widehat{\eta^2_{15}}\}$ where $\check{p}_{2_*}(\widehat{\eta^2_{15}}) = \eta^2_{15}$, equivalently speaking, $\pi_{17}(F_2) = \mathbb{Z}/8\{j_2\zeta_6\} \oplus \mathbb{Z}/2\{j_2\bar{\nu}_6\nu_{14}\} \oplus \mathbb{Z}/2\{\widehat{j_2}\widehat{\eta^2_{15}}\}$.

2. $\pi_{16}(F_2)$

We consider the fibre ation sequence,

$$\Omega S^{15} \xrightarrow{\partial'_2} J(M_{S^6}, S^{14}) \xrightarrow{\check{j}_2} S^6 \cup_{f_2} e^{15} \xrightarrow[\check{p}_2]{pinch} S^{15},$$

where $sk_{19}(J(M_{S^6}, S^{14})) = S^6$. We have the exact sequence,

$$\pi_{17}(S^{15}) \xrightarrow{\partial'_{2*17}} \pi_{16}(J(M_{S^6}, S^{14})) \longrightarrow \pi_{16}(F_2) \longrightarrow \pi_{16}(S^{15}) \xrightarrow{\partial'_{2*16}} \pi_{15}(J(M_{S^6}, S^{14}))$$

We need $\mathrm{cok}(\partial'_{2_{*17}})$ and $\mathrm{Ker}(\partial'_{2_{*16}})$.

By the computation of $\pi_{17}(F_2)$, we know $\mathrm{cok}(\partial'_{2_{*17}}) \approx \pi_{16}(J(M_{S^6}, S^{14})) \approx \pi_{16}(S^6) = \mathbb{Z}/8\{\nu_6\sigma_9\} \oplus \mathbb{Z}/2\{\eta_6\mu_7\}$. Then, we observe the commutative diagram,

$$\begin{tikzcd}
\pi_{15}(S^{14}) \arrow[r, "(2\bar{\nu}_6)_*"] \arrow[d, "\approx"'] & \pi_{15}(S^6) \arrow[d, "\approx"] \\
\pi_{16}(S^{15}) \arrow[r, "\partial'_{2*16}"] & \pi_{15}(J(M_{S^6}, S^{14}))
\end{tikzcd}$$

Since $\pi_{15}(S^{14}) \approx \mathbb{Z}/2$, we get $(2\bar{\nu}_6)_* : \pi_{16}(S^{14}) \to \pi_{16}(S^6)$ is zero. Hence, $\mathrm{Ker}(\partial'_{2_{*16}}) =$

$\pi_{16}(S^{15}) = \mathbb{Z}/2\{\eta_{15}\}$. So we have the exact sequence,

$$0 \to \mathbb{Z}/8\{\nu_6\sigma_9\} \oplus \mathbb{Z}/2\{\eta_6\mu_7\} \longrightarrow \pi_{16}(F_2) \longrightarrow \mathbb{Z}/2\{\eta_{15}\} \to 0$$

We observe the homotopy-commutative diagram with rows fibre ation sequences where $sk_{19}(J(M_{S^6}, S^{14})) = S^6$*, and* $sk_{21}(J(M_{S^7}, S^{15})) = S^7$,

$$\begin{tikzcd}
J(M_{S^6}, S^{14}) \arrow[r, "\check{j}_2"] \arrow[d, "c_2"] & S^6 \cup_{f_5} e^{15} \arrow[r, "\check{p}_2"] \arrow[d, "\Omega\Sigma"'] & S^{15} \arrow[d, "\Omega\Sigma"] \\
\Omega J(M_{S^7}, S^{15}) \arrow[r] & \Omega(S^7 \vee S^{16}) \arrow[r] & \Omega S^{16}
\end{tikzcd}$$

It induces the commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \pi_{16}(S^6) \arrow[r, "\check{j}_{2*}"] \arrow[d, "\Sigma"'] & \pi_{16}(S^6 \cup_{f_2} e^{15}) \arrow[r, "\check{p}_{2*}"] \arrow[d, "\Sigma"'] & \pi_{16}(S^{15}) \arrow[r] \arrow[d, "\Sigma"', "\approx"] & 0 \\
0 \arrow[r] & \pi_{17}(S^7) \arrow[r] & \pi_{17}(S^7) \oplus \pi_{17}(S^{16}) \arrow[r] & \pi_{17}(S^{16}) \arrow[r] & 0
\end{tikzcd}$$

According to *Page* 66 *of* [1], $\Sigma : \pi_{16}(S^6) \to \pi_{17}(S^7)$ is an isomorphism. By the Five Lemma, $\pi_{16}(F_2) \approx \pi_{17}(S^7) \oplus \pi_{17}(S^{16})$, what's more, both the two exact sequence split, we have $\pi_{16}(F_2) = \mathbb{Z}/8\{j_2\nu_6\sigma_9\} \oplus \mathbb{Z}/2\{j_2\eta_6\mu_7\} \oplus \mathbb{Z}/2\{\widehat{j_2\eta_{15}}\}$ where $\check{p}_{2_*}(\widehat{\eta_{15}}) = \eta_{15}$.

3. $\pi_{15}(F_2)$

We consider the fibre ation sequence,

$$\Omega S^{15} \xrightarrow{\partial_2'} J(M_{S^6}, S^{14}) \xrightarrow{\check{j}_2} S^6 \cup_{f_2} e^{15} \xrightarrow[\check{p}_2]{pinch} S^{15},$$

where $sk_{19}(J(M_{S^6}, S^{14})) = S^6$. We have the exact sequence,

$$\pi_{16}(S^{15}) \xrightarrow{\partial'_{2*16}} \pi_{15}(J(M_{S^6}, S^{14})) \longrightarrow \pi_{15}(F_2) \xrightarrow{\check{p}_{2*15}} \pi_{15}(S^{15}) \longrightarrow \pi_{14}(S^6)$$

We need $\mathrm{cok}(\partial'_{2_{*16}})$ and $\mathrm{Im}(\check{p}_{2_{*15}})$.

By the computation of $\pi_{16}(F_2)$, we know $\mathrm{cok}(\partial'_{2_{*16}}) \approx \pi_{15}(J(M_{S^6}, S^{14})) \approx \pi_{15}(S^6) = \mathbb{Z}/8\{\nu_6^3\} \oplus \mathbb{Z}/2\{\mu_6\} \oplus \mathbb{Z}/2\{\eta_6\varepsilon_7\}$. And obviously $\mathrm{Im}(\check{p}_{2_{*15}}) \approx \mathbb{Z}_{(2)}$. So we have the exact sequence,

$$0 \longrightarrow \mathbb{Z}/8\{\nu_6^3\} \oplus \mathbb{Z}/2\{\mu_6\} \oplus \mathbb{Z}/2\{\eta_6\varepsilon_7\} \longrightarrow \pi_{15}(F_2) \longrightarrow \mathbb{Z}_{(2)} \longrightarrow 0$$

This gives $\pi_{15}(F_2) = \mathbb{Z}/2\{j_2\nu_6^3\} \oplus \mathbb{Z}/2\{j_2\mu_6\} \oplus \mathbb{Z}/2\{j_2\eta_6\varepsilon_7\} \oplus \mathbb{Z}_{(2)}\{\breve{a}\}$ for one $\breve{a} \in \pi_{15}(F_2)$.

4. $\pi_{19}(F_4)$

By lemma 2.9.5, we have $sk_{29}(F_4) = S^8 \cup_{f_4} e^{19}$, where $f_4 = \nu_8\sigma_{11} - 2t'\sigma_8\nu_{15}$, for one odd t'. We consider the fibre sequence,

$$\Omega S^{19} \xrightarrow{\partial_4'} J(M_{S^8}, S^{18}) \xrightarrow{\breve{j}_4} S^8 \cup_{f_4} e^{19} \xrightarrow[\breve{p}_4]{pinch} S^{19}$$

By Remark 2.6.8, we have $sk_{25}(J(M_{S^8}, S^{11})) = S^8$. Then we have the exact sequence,

$$\pi_{20}(S^{19}) \xrightarrow{\partial'_{4_*20}} \pi_{19}(J(M_{S^8}, S^{18})) \longrightarrow \pi_{19}(F_4) \xrightarrow{\breve{p}_{4_*19}} \pi_{19}(S^{19}) \longrightarrow \pi_{18}(S^8)$$

We need $\mathrm{cok}(\partial'_{4_*20})$ and $\mathrm{Ker}(\partial'_{4_*19})$. Now, we observe the commutative diagram,

$$\begin{tikzcd}
 & \mathbb{Z}/8\{\zeta_8\} \oplus \mathbb{Z}/2\{\overline{\nu}_8\nu_{16}\} \arrow[d, phantom, "||"] \\
\pi_{19}(S^{18}) \arrow[r, "f_{4*}"] \arrow[d, "\approx"'] & \pi_{19}(S^8) \arrow[d, "\approx"] \\
\pi_{20}(S^{19}) \arrow[r, "\partial'_{4_*20}"] & \pi_{19}(J(M_{S^8}, S^{18}))
\end{tikzcd}$$

By *Page* 152 *of* [1], $\nu_8\sigma_{11}\eta_{18} = \nu_8\varepsilon_{11}$, by *Page* 70 *of* [1] , $\nu_8\varepsilon_{11} = 2\overline{\nu}_8\nu_{16}$. So $f_{4_*} = 0$. Thus $\mathrm{cok}(\partial'_{4_*28}) = \pi_{19}(J(M_{S^8}, S^{18})) = \pi_{19}(S^8) \approx \mathbb{Z}/8 \oplus \mathbb{Z}/2$. Next, we observe the commutative diagram,

$$\begin{tikzcd}
 & \mathbb{Z}/8\{f_4\} \oplus else \arrow[d, phantom, "||"] \\
\pi_{18}(S^{18}) \arrow[r, "f_{4*}"] \arrow[d, "\approx"'] & \pi_{18}(S^8) \arrow[d, "id"] \\
\pi_{19}(S^{19}) \arrow[r, "\partial'_{4_*20}"] & \pi_{18}(S^{18})
\end{tikzcd}$$

Here we uesd the identification $\pi_{18}(J(M_{S^8}, S^{11})) = \pi_{18}(S^8)$. So, $\mathrm{Ker}(\partial'_{4_*19}) = \mathbb{Z}_{(2)}\{8\iota_{19}\}$.

Then we have the exact sequence,

$$0 \longrightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/2 \longrightarrow \pi_{19}(F_4) \longrightarrow \mathbb{Z}_{(2)} \longrightarrow 0$$

This gives $\pi_{19}(F_4) \approx \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)}$, and then it's easy to get that $\pi_{19}(F_4) = \mathbb{Z}/8\{j_4\zeta_8\} \oplus \mathbb{Z}/2\{j_4\overline{\nu}_8\nu_{16}\} \oplus \mathbb{Z}_{(2)}\{\acute{a}\}$ for one $\acute{a} \in \pi_{19}(F_4)$ satisfying $\check{p}_{4_{*19}}(\acute{a}) = 8\iota_{19}$.

5.$\pi_{20}(F_5)$

By lemma 2.9.5, $sk_{32}(F_5) = S^9 \cup_{f_5} e^{21}$, $f_5 = \overline{\nu}_9\nu_{17}$. Because $\pi_{20}(S^9) = \mathbb{Z}/8\{\zeta_9\} \oplus \mathbb{Z}/2\{\overline{\nu}_9\nu_{17}\}$. Hence $j_{5_*}(\overline{\nu}_9\nu_{17}) = 0$, $\pi_{20}(F_5) = \mathbb{Z}/8\{j_5\zeta_9\}$ $\square$.

Lemma 4.7 After localization at 2, for $k \geq 0$ and a CW complex X, suppose that $\widetilde{H}_*(X) \approx \widetilde{H}_*(\Sigma^k\mathbb{H}P^2)$ as modules over Steenord algebra, then $X \simeq \Sigma^k\mathbb{H}P^2$ localized at 2.

Proof

For $k = 0$, this is got by *Lemma* 3.9 *of* [8] immediately.

For $k \geq 1$, we notice that $\pi_{7+k}(S^{4+k}) = \mathbb{Z}/8\{\nu_{4+k}\}$. To obtain a contradiction, suppose $X = S^{4+k} \cup_{2^n\nu_{4+k}} e^{8+k}$, $n = 2$ *or* 4. Now, we observe the homotopy-commutative diagram with rows cofibre sequences,

$$\begin{tikzcd}
S^{7+k} \arrow[r, "\nu_{4+k}"] \arrow[d, "id"'] & S^{4+k} \arrow[r] \arrow[d, "{[2^n]}"'] & \Sigma^k\mathbb{H}P^2 \arrow[r] \arrow[d, "\varphi"'] & S^{8+k} \arrow[r] \arrow[d, "id"'] & \cdots \\
S^{7+k} \arrow[r, "2^n\nu_{4+k}"] & S^{4+k} \arrow[r] & X \arrow[r] & S^{8+k} \arrow[r] & \cdots
\end{tikzcd}$$

Let $\widetilde{H}_*(\Sigma^k\mathbb{H}P^2) = \mathbb{Z}/2\{a_{4+k}, a_{8+k}\}$, $\widetilde{H}_*(X) = \mathbb{Z}/2\{x_{4+k}, x_{8+k}\}$, where $|a_i| = i$, $|x_i| = i$, and $Sq^4_*(a_{8+k}) = a_{4+k}$. Consider the diagram of mod 2 reduced homology groups induced by the above diagram, we have $\varphi_*(a_{8+k}) = x_{8+k}$, and $\varphi_*(a_{4+k}) = 2^n x_{4+k} = 0$. Thus, $Sq^4_*(x_{8+k}) = 0$, while $\widetilde{H}_*(X) \approx \widetilde{H}_*(\Sigma^k\mathbb{H}P^2)$ as modules over Steenord algebra , impossible. This forces that $X \simeq \Sigma^k\mathbb{H}P^2$. $\square$

Theorem 4.8 After localization at 2,

$$\pi_{7+k}(\Sigma^k\mathbb{H}P^2) \approx \begin{cases} 0, & k \geq 1, \\ \mathbb{Z}/4, & k = 0 \end{cases}$$

$$\pi_{8+k}(\Sigma^k\mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}_{(2)}, & k \geq 1, \\ \mathbb{Z}/2, & k = 0 \end{cases}$$

$$\pi_{9+k}(\Sigma^k\mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/2, & k \geq 3 \text{ or } k = 0, \\ \mathbb{Z}_{(2)}\ , & k = 2, \\ 0, & k = 1 \end{cases}$$

$$\pi_{10+k}(\Sigma^k\mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/2, & k \geq 1, \\ 0, & k = 0 \end{cases}$$

Proof

1. $\pi_{7+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{7+k}(\Sigma^k\mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k \geq 7 - 6 = 1$

For $k \geq 1$, by lemma 4.1, $\pi_{7+k}(\Sigma^k\mathbb{H}P^2) \approx \pi_7^s(\mathbb{H}P^\infty) = 0$.

For $k = 0$, $\pi_7(\mathbb{H}P^2) \approx \pi_6(S^3) \approx \mathbb{Z}/4$.

2. $\pi_{8+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{8+k}(\Sigma^k\mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k \geq 8 - 6 = 2$.

For $k \geq 2$, by lemma 4.1, $\pi_{8+k}(\Sigma^k\mathbb{H}P^2) \approx \pi_8^s(\mathbb{H}P^\infty) = \mathbb{Z}_{(2)}$.

For $k = 1$, we observe the exact sequence,

$$\pi_{10}(S^9) \xrightarrow{\partial_{1_{*10}}} \pi_9(F_1) \longrightarrow \pi_9(\Sigma\mathbb{H}P^2) \xrightarrow{p_{1_{*9}}} \pi_9(S^9) \longrightarrow \pi_8(S^6)$$

Obviously, $\mathrm{Im}(p_{1_{*9}}) \approx \mathbb{Z}_{(2)}$. We need $\mathrm{cok}(\partial_{1_{*10}})$. We observe the commutative diagram,

$$\begin{array}{ccc} \mathbb{Z}/2\{\eta_8\} & & \mathbb{Z}/2\{\nu_5\eta_8\} \\ \| & & \| \\ \pi_9(S^8) & \xrightarrow{\nu_{5_*}} & \pi_9(S^5) \\ \approx\downarrow & & \downarrow\approx \\ \pi_{10}(S^9) & \xrightarrow{\partial_{1_{*10}}} & \pi_9(F_1) \\ \| & & \| \\ \mathbb{Z}/2\{\eta_9\} & & \mathbb{Z}/2\{j_1\mu_5\eta_8\} \end{array}$$

Thus $\operatorname{cok}(\partial_{1_{*10}})=0$. Then, we have the exact aequence,

$$0 \to 0 \longrightarrow \pi_9(\Sigma\mathbb{H}P^2) \longrightarrow \mathbb{Z}_{(2)} \to 0$$

So, $\pi_9(\Sigma\mathbb{H}P^2)\approx\mathbb{Z}_{(2)}$. In fact, the result can be stronger, that is , $\Sigma:\pi_9(\Sigma\mathbb{H}P^2)\to\pi_{10}(\Sigma^2\mathbb{H}P^2)$ is an isomorphism . Since we have the following commutative diagram with exact rows (the left groups are replaced by the cokernels).

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_9(\Sigma\mathbb{H}P^2) \arrow[r] \arrow[d, "\Sigma"'] & \pi_9(S^9) \arrow[r] \arrow[d, "\approx"] & \pi_8(S^5) \arrow[d, "\approx"] \\
0 \arrow[r] & 0 \arrow[r] & \pi_{10}(\Sigma^2\mathbb{H}P^2) \arrow[r] & \pi_{10}(S^{10}) \arrow[r] & \pi_9(S^6)
\end{tikzcd}$$

For $k=0$, $\pi_8(\mathbb{H}P^2)\approx\pi_7(S^3)\approx\mathbb{Z}/2$.

3. $\pi_{9+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{9+k}(\Sigma^k\mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k\geq 9-6=3$

For $k\geq 3$, by Lemma 4.1, $\pi_{9+k}(\Sigma^k\mathbb{H}P^2)\approx\pi_9^s(\mathbb{H}P^\infty)\approx\mathbb{Z}/2$.

For $k=1$, we need $\operatorname{cok}(\partial_{1_{*11}})$ and $\operatorname{Ker}(\partial_{1_{*10}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/16\{\eta_8^2\} \arrow[d, equal] & \mathbb{Z}/2\{\nu_5\eta_8^2\} \arrow[d, equal] \\
\pi_{10}(S^8) \arrow[r, "\nu_{5*}"] \arrow[d, "\approx"'] & \pi_{10}(S^5) \arrow[d, "\approx"] \\
\pi_{11}(S^9) \arrow[r, "\partial_{1_{*11}}"] & \pi_{10}(F_1)
\end{tikzcd}$$

Thus $\operatorname{cok}(\partial_{1_{*11}})=0$. Next we observe the commutative diagram,

$$\begin{array}{ccc}
\mathbb{Z}/2\{\eta_8\} & & \mathbb{Z}/2\{\nu_5\eta_8\} \\
\| & & \| \\
\pi_9(S^8) & \xrightarrow{\nu_{5*}} & \pi_9(S^5) \\
\downarrow\approx & & \downarrow\approx \\
\pi_{10}(S^9) & \xrightarrow{\partial_{1_{*10}}} & \pi_9(F_1)
\end{array}$$

So, $\mathrm{Ker}(\partial_{1_{*10}}) = 0$. Then, we have the exact aequence,

$$0 \to 0 \longrightarrow \pi_{10}(\Sigma\mathbb{H}P^2) \longrightarrow 0 \to 0$$

This gives $\pi_{10}(\Sigma\mathbb{H}P^2) = 0$.

For $k = 2$, we need $\mathrm{cok}(\partial_{2_{*12}})$ and $\mathrm{Ker}(\partial_{1_{*11}})$. Since $\partial_{2_{*12}} : \pi_{12}(S^{10}) \to \pi_{11}(F_2) \approx \mathbb{Z}_{(2)}$, $\partial_{2_{*11}} : \pi_{11}(S^{10}) \to \pi_{10}(F_2) = 0$, thus $\mathrm{cok}(\partial_{2_{*12}}) \approx \mathbb{Z}_{(2)}$, $\mathrm{Ker}(\partial_{2_{*11}}) \approx \mathbb{Z}/2$. Then, we have the exact aequence,

$$0 \to \mathbb{Z}_{(2)} \longrightarrow \pi_{21}(\Sigma^7\mathbb{H}P^2) \longrightarrow \mathbb{Z}/2 \to 0$$

This gives $\pi_{11}(\Sigma^2\mathbb{H}P^2) \approx \mathbb{Z}_{(2)}$ *or* $\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$.

While $S^6 \xrightarrow{i} \Sigma^2\mathbb{H}P^2 \to (\Sigma^2\mathbb{H}P^2, S^6)$ induces the exact sequence

$$\begin{array}{ccccccccc}
0 \longrightarrow & \pi_{11}(S^6) & \xrightarrow{i_*} & \pi_{11}(\Sigma^2\mathbb{H}P^2) & \longrightarrow & \pi_{10}(S^9) & \longrightarrow 0 \\
& \| & & & & \| & \\
& \mathbb{Z}_{(2)}\{[\iota_6, \iota_6]\} & & & & \mathbb{Z}/2\{\eta_9\} &
\end{array}$$

Since $\nu_6\eta_9 \in \pi_{10}(S^6) = 0$, $[\iota_6, \iota_6] \in \pm\{\nu_6, \eta_9, 2\iota_{10}\}$, so $i_*([\iota_6, \iota_6]) \in \pm i \circ \{\nu_6, \eta_9, 2\iota_{10}\} = \mp 2\{i, \nu_6, \eta_9\}$. Thus, $i_*([\iota_6, \iota_6])$ can be divisible by 2. Then, $\pi_{11}(\Sigma^2\mathbb{H}P^2) \approx \mathbb{Z}_{(2)}$.

4. $\pi_{10+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{10+k}(\Sigma^k\mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k \geq 10 - 6 = 4$

For $k \geq 4$, by Lemma 4.1, $\pi_{10+k}(\Sigma^k\mathbb{H}P^2) \approx \pi_{10+k}(\Sigma^k\mathbb{H}P^\infty) \approx \pi^s_{10}(\mathbb{H}P^\infty) \approx \mathbb{Z}/2$.

For $1 \leq k \leq 3$, we observe the homotopy-commutative diagram with rows fibre sequences,

$$\begin{tikzcd}
F_k \arrow[r, "i_k"] \arrow[d, "\varphi_{k+(4-k)}"] & \Sigma^k\mathbb{H}P^2 \arrow[r, "p_k"] \arrow[d, "w_{k+(4-k)}"] & S^{8+k} \arrow[d, "\theta_{k+(4-k)}"] \\
\Omega^{4-k}F_4 \arrow[r, "\Omega^{4-k}i_4"] & \Omega^{4-k}\Sigma^4\mathbb{H}P^2 \arrow[r, "\Omega^{4-k}p_4"] & \Omega^{4-k}S^{12}
\end{tikzcd}$$

it induces the commutative diagram with the exact rows ,

$$\begin{tikzcd}
\pi_{11+k}(S^{8+k}) \arrow[r] \arrow[d, "\approx"] & \pi_{10+k}(F_k) \arrow[r] \arrow[d, "\varphi_{k+(4-k)_{*(10+k)}}"] & \pi_{10+k}(\Sigma^k\mathbb{H}P^2) \arrow[r] \arrow[d, "w_{k+(4-k)_*}"] & \pi_{10+k}(S^{8+k}) \arrow[r] \arrow[d, "\approx"] & \pi_{9+k}(F_k) \arrow[d, "\varphi_{k+(4-k)_{*(9+k)}}"] \\
\pi_{15}(S^{12}) \arrow[r] & \pi_{14}(F_4) \arrow[r] & \pi_{14}(\Sigma^4\mathbb{H}P^2) \arrow[r] & \pi_{14}(S^{12}) \arrow[r] & \pi_{13}(F_4)
\end{tikzcd}$$

By lemma 3.4, $\varphi_{k+(4-k)_{*(10+k)}}$ and $\varphi_{k+(4-k)_{*(9+k)}}$ are isomorphisms for $1 \leq k \leq 3$. According to the Five Lemma, $w_{k+(4-k)_*} : \pi_{10+k}(\Sigma^k\mathbb{H}P^2) \to \pi_{15}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}/2$ are isomorphisms for $1 \leq k \leq 3$.

For $k = 0$, $\pi_{10}(\mathbb{H}P^2) \approx \pi_{10}(\mathbb{H}P^\infty) \approx \pi_9(S^3) = 0$. □

Theorem 4.9 After localization at 2,

$$\pi_{11+k}(\Sigma^k\mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/16 \oplus \mathbb{Z}/4, & k \geq 5 \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}_{(2)}, & k = 4 \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4, & k = 3 \\ \mathbb{Z}/8 \oplus \mathbb{Z}/2, & k = 2 \\ \mathbb{Z}/8, & k = 1 \\ \mathbb{Z}_{(2)} \ , & k = 0 \end{cases}$$

$$\pi_{12+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}(\mathbb{Z}/2)^2, & k\geq 6 \ or\ k=0\\ (\mathbb{Z}/2)^3, & k=5 \ or\ 3\\ (\mathbb{Z}/2)^4, & k=4,\\ \mathbb{Z}/2, & k=2,\\ \mathbb{Z}_{(2)}\oplus\mathbb{Z}/2, & k=1\end{cases}$$

$$\pi_{13+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}(\mathbb{Z}/2)^2, & k\geq 7\\ (\mathbb{Z}/2)^2\oplus\mathbb{Z}_{(2)}, & k=6 \ or\ 2\\ (\mathbb{Z}/2)^3, & k=5,\ 3,\ 1\ or\ 0\\ (\mathbb{Z}/2)^4, & k=4\end{cases}$$

Proof

1. $\pi_{11+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{11+k}(\Sigma^k\mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k\geq 11-6=5$

For $k\geq 5$, according to Lemma 4.1, $\pi_{11+k}(\Sigma^k\mathbb{H}P^2)\approx\mathbb{Z}/4\oplus\mathbb{Z}/16$.

For $k=3$, since $sk_{16}(F_3)=S^7$, $sk_{20}(F_5)=S^9$, similarly as the computation of $Ker(\partial_{1*_{10}})$, we have $Ker(\partial_{3*_{14}})=\mathbb{Z}/4\{2\nu_{11}\}$.

Therefore, we have the exact sequence,

$$0\to\mathbb{Z}/8\{j_3'\sigma'\}\to\pi_{14}(\Sigma^3\mathbb{H}P^2)\to\mathbb{Z}/4\{2\nu_{11}\}\to 0\ ,$$

this gives $\pi_{14}(\Sigma^3\mathbb{H}P^2)\approx\mathbb{Z}/32\ or\ \mathbb{Z}/8\oplus\mathbb{Z}/4\ or\ \mathbb{Z}/16\oplus\mathbb{Z}/2$.

Since $p_{3_*}(\{j_3',\nu_7,2\nu_{10}\})=p_3\circ\{j_3',\nu_7,2\nu_{10}\}=-\{p_3,j_3',\nu_7\}\circ 2\nu_{11}\ni -2\nu_{11}$,

so $\exists x\in\{j_3',\nu_7,2\nu_{10}\}$ such that $p_{3_*}(x)=-2\nu_{11}$. Then $ord(x)\geq 4$.

While $4x\in\{j_3',\nu_7,2\nu_{10}\}\circ 4\iota_{14}=-j_3'\circ\{\nu_7,2\nu_{10},4\iota_{13}\}=\mathbb{Z}/2\{4j_3'\sigma'\}$ (Notice that $4\sigma'\in\{\nu_7,2\nu_{10},4\iota_{13}\}$ and $j'_{3_{*14}}$ is into). So $ord(4x)=2$ or 1. Combine with $ord(x)\geq 4$, we know $ord(x)=8$ or 4. It obviously forces that $\pi_{14}(\Sigma^3\mathbb{H}P^2)$ cannot be isomorphic to $\mathbb{Z}/32$. We notice that if $\pi_{14}(\Sigma^3\mathbb{H}P^2)\approx\mathbb{Z}/16\oplus\mathbb{Z}/2$, while $p_{3_*}(x)$ generates $\mathrm{Im}(p_{3*})$, so $ord(x)=16$, impossible. So $\pi_{14}(\Sigma^3\mathbb{H}P^2)$ cannot be isomorphic to $\mathbb{Z}/16\oplus\mathbb{Z}/2$. In conclusion, $\pi_{14}(\Sigma^3\mathbb{H}P^2)\approx\mathbb{Z}/8\oplus\mathbb{Z}/4$. Precisely speaking, $\pi_{14}(\Sigma^3\mathbb{H}P^2)=\mathbb{Z}/8\{j_3\sigma'\}\oplus\mathbb{Z}/4\{x\}$, for one $x\in\{j_3',\nu_7,2\nu_{10}\}$.

For $k = 4$, since $sk_{18}(F_4) = S^8$, similarly as above, $\mathrm{Im}(p_{4*})= \mathrm{Ker}(\partial_{4*_{15}}) = \mathbb{Z}/4\{2\nu_{12}\}$, then we have the exact sequence,

$$0 \to \mathbb{Z}_{(2)}\{\sigma_8\} \oplus \mathbb{Z}/8\{\Sigma\sigma'\} \to \pi_{15}(\Sigma^4\mathbb{H}P^2) \to \mathbb{Z}/4\{2\nu_{12}\} \to 0\ ,$$

by Lemma 2.2.2, we have $\pi_{15}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus A$,
where $A = \mathbb{Z}/8,\ \mathbb{Z}/16,\ \mathbb{Z}/8 \oplus \mathbb{Z}/2,\ \mathbb{Z}/32\ or\ \mathbb{Z}/8 \oplus \mathbb{Z}/4$.
Comparing with the sequence of $\pi_{14}(\Sigma^3\mathbb{H}P^2)$,we observe the commutative diagram with short exact rows,

$$\begin{tikzcd}
\mathbb{Z}/8 \arrow[r] \arrow[d, "into"] & \mathbb{Z}/8 \oplus \mathbb{Z}/4 \arrow[r] \arrow[d, "w_{3+1*}"] & \mathbb{Z}/4 \arrow[d, "\approx"] \\
\mathbb{Z}_{(2)} \oplus \mathbb{Z}/8 \arrow[r] & \pi_{15}(\Sigma^4\mathbb{H}P^2) \arrow[r] & \mathbb{Z}/4
\end{tikzcd}$$

By the Snake Lemma, w_{3+1*} is into, so $\pi_{15}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/4$.

For $k = 2$,similarly as the computation of $\mathrm{Ker}(\partial_{1*_{10}})$, we have $\mathrm{Ker}(\partial_{2*_{13}}) = \mathbb{Z}/4\{2\nu_{10}\}$. Therefore, we have the exact sequence,

$$0 \to \mathbb{Z}/4\{j_2'\sigma''\} \to \pi_{13}(\Sigma^2\mathbb{H}P^2) \to \mathbb{Z}/4\{2\nu_{10}\} \to 0\ ,$$

this gives $\pi_{13}(\Sigma^2\mathbb{H}P^2) \approx \mathbb{Z}/16\ or\ \mathbb{Z}/8 \oplus \mathbb{Z}/2\ or\ \mathbb{Z}/4 \oplus \mathbb{Z}/4$.
Since $p_{2_*}(\{j_2', \nu_6, 2\nu_9\}) = p_2 \circ \{j_2', \nu_6, 2\nu_9\} = -\{p_2, j_2', \nu_6\} \circ 2\nu_{10} \ni -2\nu_{10}$,
so $\exists z \in \{j_2', \nu_6, 2\nu_9\}$ such that $p_{2_*}(z) = -2\nu_{10}$.Then $ord(z) \geq 4$.
While $4z \in \{j_2', \nu_6, 2\nu_9\} \circ 4\iota_{13} = -j_2' \circ \{\nu_6, 2\nu_9, 4\iota_{12}\}\ mod\ 0$,and $-2\sigma'' \in \{\nu_6, 2\nu_9, 4\iota_{12}\}$, so $4z = 2j_2'\sigma''$ is of order 2. Then $ord(z) = 8$. Comparing with the sequence of $\pi_{13}(\Sigma^2\mathbb{H}P^2)$,we observe the commutative diagram with short exact rows,

$$\begin{tikzcd}
\mathbb{Z}/4 \arrow[r] \arrow[d, "into"] & \pi_{13}(\Sigma^2\mathbb{H}P^2) \arrow[r] \arrow[d, "w_{2+1*}"] & \mathbb{Z}/4 \arrow[d, "\approx"] \\
\mathbb{Z}/8 \arrow[r] & \mathbb{Z}/8 \oplus \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/4
\end{tikzcd}$$

By the Snake Lemma, w_{2+1*} is into, while $ord(z) = 8$, so $\pi_{13}(\Sigma^2\mathbb{H}P^2) \approx \mathbb{Z}/8 \oplus \mathbb{Z}/2$.

For $k = 1$,similarly as the computation of $\mathrm{Ker}(\partial_{1*_{10}})$, we have $\mathrm{Ker}(\partial_{1*_{12}}) = \mathbb{Z}/4\{2\nu_9\}$. Therefore, we have the exact sequence,

$$0 \to \mathbb{Z}/2\{j_1'\sigma'''\} \to \pi_{12}(\Sigma\mathbb{H}P^2) \to \mathbb{Z}/4\{2\nu_9\} \to 0\ ,$$

this gives $\pi_{12}(\Sigma^2\mathbb{H}P^2) \approx \mathbb{Z}/8\ or\ \mathbb{Z}/2 \oplus \mathbb{Z}/4$.

Since $p_{1_*}(\{j_1', \nu_5, 2\nu_8\}) = p_1 \circ \{j_1', \nu_5, 2\nu_8\} = -\{p_1, j_1', \nu_5\} \circ 2\nu_9 \ni -2\nu_9$,
so $\exists x' \in \{j_1', \nu_5, 2\nu_8\}$ such that $p_{1_*}(x') = -2\nu_9$.Then $ord(x') \geq 4$.
While $4x' \in \{j_1', \nu_5, 2\nu_8\} \circ 4\iota_{11} = -j_1' \circ \{\nu_5, 2\nu_8, 4\iota_{10}\} = \{j_1'\sigma'''\}$ is of order 2 . Then $ord(x') = 8$, so $\pi_{12}(\Sigma\mathbb{H}P^2) \approx \mathbb{Z}/8$.

For $k = 0$, $\pi_{12}(\mathbb{H}P^2) \approx \pi_{12}(S^{11}) \oplus \pi_{11}(S^3) \approx (\mathbb{Z}/2)^2$. $\square$

3. $\pi_{13+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{13+k}(\Sigma^k\mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k \geq 13 - 6 = 7$

For $k \geq 7$, according to Lemma 4.1, $\pi_{13+k}(\Sigma^k\mathbb{H}P^2) \approx (\mathbb{Z}/2)^2$.

For $1 \leq k \leq 6$ but $k \neq 2$, we observe the homotopy-commutative diagram with rows fibre sequences,

$$\begin{tikzcd}
F_k \arrow[r, "i_k"] \arrow[d, "\varphi_{k+(7-k)}"] & \Sigma^k\mathbb{H}P^2 \arrow[r, "p_k"] \arrow[d, "w_{k+(7-k)}"] & S^{8+k} \arrow[d, "\theta_{k+(7-k)}"] \\
\Omega^{7-k}F_4 \arrow[r, "\Omega^{7-k}i_7"] & \Omega^{7-k}\Sigma^4\mathbb{H}P^2 \arrow[r, "\Omega^{7-k}p_7"] & \Omega^{7-k}S^{12}
\end{tikzcd}$$

it induces the commutative diagram with the exact rows ,

$$\begin{tikzcd}
\pi_{14+k}(S^{8+k}) \arrow[r, "\partial_{k_*}"] \arrow[d, "(\approx)\ \theta_{k+(7-k)_*}"] & \pi_{13+k}(F_k) \arrow[r, "i_{k_*}"] \arrow[d, "\varphi_{k+(7-k)_*}"] & \pi_{13+k}(\Sigma^k\mathbb{H}P^2) \arrow[r] \arrow[d] & 0 \\
\pi_{21}(S^{15}) \arrow[r, "\partial_{7_*}"] \arrow[d, equal] & \pi_{20}(F_7) \arrow[r, "i_{7_*}"] \arrow[d, equal] & \pi_{20}(\Sigma^7\mathbb{H}P^2) \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z}/2 & (\mathbb{Z}/2)^3 & (\mathbb{Z}/2)^2 &
\end{tikzcd}$$

So $\mathrm{Im}(i_{7_*}) \approx (\mathbb{Z}/2)^2$, successively , $|Ker(i_{7_*})| = \frac{|(\mathbb{Z}/2)^3|}{|(\mathbb{Z}/2)^2|} = 2$. Hence, $\mathrm{Ker}(i_{7_*}) \approx \mathbb{Z}/2$, thus, $\mathrm{Im}(\partial_{7_*}) \approx \mathbb{Z}/2$, $\partial_{7_*} : \pi_{21}(S^{15}) \to \pi_{20}(F_7)$ is into. Since $\pi_{14+k}(S^{8+k}) \approx \mathbb{Z}/2$, we have $\mathrm{Im}(\partial_{k_*}) \approx \mathbb{Z}/2$ *or* 0. While $\varphi_{k+(7-k)_*}\partial_{k_*} = \partial_{7_*}\theta_{k+(7-k)_*}$ is not trivial, so $\mathrm{Im}(\partial_{k_*}) \neq 0$. This gives $Im(\partial_{k_*}) \approx \mathbb{Z}/2$. We know that $\pi_{13+k}(\Sigma^k\mathbb{H}P^2)$ =$\mathrm{Im}(i_{k_*}) \approx \pi_{13+k}(F_k)/\mathrm{Im}(\partial_{k_*})$. By using $sk_{10+2k}(F_k) = S^{4+k}$, $sk_{20}(F_1) = S^5 \vee S^{13}$ and $sk_{26}(F_3) = S^7 \vee S^{17}$, we get the results .

For $k=2$, we need $\mathrm{cok}(\partial_{2_{*16}})$ and $\mathrm{Ker}(\partial_{2_{*15}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/2\{\nu_9^2\} \arrow[d, equal] & \mathbb{Z}/2\{\nu_6^3\}\oplus\mathbb{Z}/2\{\mu_6\}\oplus\mathbb{Z}/2\{\eta_6\varepsilon_7\} \arrow[d, equal] \\
\pi_{15}(S^9) \arrow[r, "\nu_{6*}"] \arrow[d, "\approx"'] & \pi_{15}(S^6) \arrow[d, "j_{2*}"] \\
\pi_{16}(S^{10}) \arrow[r, "\partial_{2*16}"] \arrow[d, equal] & \pi_{15}(F_2) \arrow[d, equal] \\
\mathbb{Z}/2\{\nu_{10}^2\} & \mathbb{Z}/2\{j_2\nu_6^3\}\oplus\mathbb{Z}/2\{j_2\mu_6\}\oplus\mathbb{Z}/2\{j_2\eta_6\varepsilon_7\}\oplus\mathbb{Z}_{(2)}\{\breve{a}\}
\end{tikzcd}$$

Thus $\mathrm{cok}(\partial_{2_{*16}})\approx\mathbb{Z}_{(2)}\oplus(\mathbb{Z}/2)^2$ .Since $\partial_{7_{*21}}:\pi_{15}(S^{10})=0\to\pi_{14}(S^6)$, we get $\mathrm{Ker}(\partial_{2_{*15}})=0$. Then, we have the exact aequence,

$$0\to\mathbb{Z}_{(2)}\oplus(\mathbb{Z}/2)^2\longrightarrow\pi_{15}(\Sigma^2\mathbb{H}P^2)\longrightarrow 0\to 0$$

This gives $\pi_{15}(\Sigma^2\mathbb{H}P^2)\approx\mathbb{Z}_{(2)}\oplus(\mathbb{Z}/2)^2$.

For $k=0$, $\pi_{13}(\mathbb{H}P^2)\approx\pi_{13}(S^{11})\oplus\pi_{12}(S^3)\approx(\mathbb{Z}/2)^3$.

Theorem 4.10 After localization at 2,

$$\pi_{14+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}\mathbb{Z}/2, & k\geq 8 \textit{ or } k=4\\ (\mathbb{Z}/2)^2, & k=7,\ 6\ ,5,\ 2 \textit{ or } 1\\ \mathbb{Z}/2\oplus\mathbb{Z}_{(2)}, & k=3,\\ \mathbb{Z}/8\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2, & k=0\end{cases}$$

$$\pi_{15+k}(\Sigma^k\mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/128, & k \geq 7 \ but\ k \neq 8 \\ \mathbb{Z}/128 \oplus \mathbb{Z}_{(2)}, & k = 8 \\ \mathbb{Z}/64, & k = 6, \\ \mathbb{Z}/32, & k = 5, \\ \mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/2, & k = 4, \\ \mathbb{Z}/16 \oplus (\mathbb{Z}/2)^2, & k = 3 \ or\ 2 \\ \mathbb{Z}/16 \oplus \mathbb{Z}/8, & k = 1, \\ (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4, & k = 0 \end{cases}$$

Proof

1. $\pi_{14+k}(\Sigma^k\mathbb{H}P^2)$

$\pi_{14+k}(\Sigma^k\mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k \geq 14 - 6 = 8$

For $k \geq 8$, by Lemma 4.1, $\pi_{14+k}(\Sigma^k\mathbb{H}P^2) \approx \mathbb{Z}/2$.

For $k = 7$, we need $\mathrm{cok}(\partial_{7_{*22}})$ and $\mathrm{Ker}(\partial_{7_{*21}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/16\{\sigma_{14}\} \arrow[d, equal] & \mathbb{Z}/2\{\sigma_{11}\nu_{18}\} \oplus \mathbb{Z}/2\{\eta_{11}\mu_{12}\} \arrow[d, equal] \\
\pi_{21}(S^{14}) \arrow[r, "\nu_{11*}"] \arrow[d, "\approx"'] & \pi_{21}(S^{11}) \arrow[d, "\approx"] \\
\pi_{22}(S^{15}) \arrow[r, "\partial_{7_{*22}}"] \arrow[d, equal] & \pi_{21}(F_7) \arrow[d, equal] \\
\mathbb{Z}/16\{\sigma_{15}\} & \mathbb{Z}/2\{j_7\sigma_{11}\nu_{18}\} \oplus \mathbb{Z}/2\{j_7\eta_{11}\mu_{12}\}
\end{tikzcd}$$

By *Page* 72 *of* [1], we have $\nu_{11}\sigma_{14} = 0$, thus $\mathrm{cok}(\partial_{7_{*22}}) \approx \mathbb{Z}/2\{\sigma_{11}\nu_{18}\} \oplus \mathbb{Z}/2\{\eta_{11}\mu_{12}\}$.
Next we observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/2\{\nu_{14}^2\} \arrow[d, equal] & \mathbb{Z}/2\{\nu_{11}^3\} \oplus else \arrow[d, equal] \\
\pi_{20}(S^{14}) \arrow[r, "\nu_{11*}"] \arrow[d, "\approx"'] & \pi_{20}(S^{11}) \arrow[d, "\approx"] \\
\pi_{21}(S^{15}) \arrow[r, "\partial_{7_{*21}}"] \arrow[d, equal] & \pi_{20}(F_7) \\
\mathbb{Z}/2\{\nu_{15}^2\} &
\end{tikzcd}$$

So, $\mathrm{Ker}(\partial_{7_{*21}}) = 0$. Then, we have the exact aequence,

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \pi_{21}(\Sigma^7 \mathbb{H}P^2) \longrightarrow 0 \to 0$$

This gives $\pi_{21}(\Sigma^7 \mathbb{H}P^2) \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

For $k = 6$, we need $\mathrm{cok}(\partial_{6_{*21}})$ and $\mathrm{Ker}(\partial_{6_{*20}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/16\{\sigma_{13}\} \arrow[d, equal] & \mathbb{Z}/4\{\sigma_{10}\nu_{17}\} \oplus \mathbb{Z}/2\{\eta_{10}\mu_{11}\} \arrow[d, equal] \\
\pi_{20}(S^{13}) \arrow[r, "\nu_{10*}"] \arrow[d, "\approx"'] & \pi_{20}(S^{10}) \arrow[d, "\approx"] \\
\pi_{21}(S^{14}) \arrow[r, "\partial_{6_{*21}}"] & \pi_{20}(F_6)
\end{tikzcd}$$

By *Page* 72 *of* [1], we have $\nu_{10}\sigma_{13} = 2t\sigma_{10}\nu_{17}$ for an odd t, thus $\mathrm{cok}(\partial_{6_{*21}}) = \mathbb{Z}/2\{cls(j_6\sigma_{10}\nu_{17})\} \oplus \mathbb{Z}/2\{j_6\eta_{10}\mu_{11}\}$, where $cls(j_6\sigma_{10}\nu_{17}) = \sigma_{10}\nu_{17} + \langle 2j_6\sigma_{10}\nu_{17}\rangle$. Next we observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/2\{\nu_{13}^2\} \arrow[d, equal] & \mathbb{Z}/2\{\nu_{10}^3\} \oplus else \arrow[d, equal] \\
\pi_{19}(S^{13}) \arrow[r, "\nu_{10*}"] \arrow[d, "\approx"'] & \pi_{19}(S^{10}) \arrow[d, "\approx"] \\
\pi_{20}(S^{14}) \arrow[r, "\partial_{6_{*20}}"] \arrow[d, equal] & \pi_{19}(F_6) \\
\mathbb{Z}/2\{\nu_{14}^2\} &
\end{tikzcd}$$

So, $\mathrm{Ker}(\partial_{6_{*20}}) = 0$. Then, we have the exact aequence,

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \pi_{20}(\Sigma^6 \mathbb{H}P^2) \longrightarrow 0 \to 0$$

This gives $\pi_{20}(\Sigma^6 \mathbb{H}P^2) \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

For $k = 5$, we need $\mathrm{cok}(\partial_{5_{*21}})$ and $\mathrm{Ker}(\partial_{5_{*20}})$. We observe the commutative diagram,

$$\begin{array}{ccc}
\mathbb{Z}/16\{\sigma_{12}\} & & \mathbb{Z}/8\{\sigma_9\nu_{16}\}\oplus\mathbb{Z}/2\{\eta_9\mu_{10}\} \\
\parallel & & \parallel \\
\pi_{19}(S^{12}) & \xrightarrow{\nu_{9*}} & \pi_{19}(S^9) \\
\approx\downarrow & & \downarrow\approx \\
\pi_{20}(S^{13}) & \xrightarrow{\partial_{5_{*20}}} & \pi_{19}(F_5) \\
\parallel & & \parallel \\
\mathbb{Z}/16\{\sigma_{13}\} & & \mathbb{Z}/8\{j_5\sigma_9\nu_{16}\}\oplus\mathbb{Z}/2\{j_5\eta_9\mu_{10}\}
\end{array}$$

By *Page* 72 *of* [1], we have $\nu_9\sigma_{12} = 2t\sigma_9\nu_{16}$ for an odd t, thus $\text{cok}(\partial_{5_{*20}}) = \mathbb{Z}/2\{cls(j_5\sigma_{10}\nu_{17})\}\oplus \mathbb{Z}/2\{j_5\eta_9\mu_{10}\}$, where $cls(j_5\sigma_9\nu_{16}) = \sigma_9\nu_{16} + \langle 2j_5\sigma_9\nu_{16}\rangle$. Next we observe the commutative diagram,

$$\begin{array}{ccc}
\mathbb{Z}/2\{\nu_{12}^2\} & & \mathbb{Z}/2\{\nu_9^3\}\oplus else \\
\parallel & & \parallel \\
\pi_{18}(S^{12}) & \xrightarrow{\nu_{9*}} & \pi_{18}(S^9) \\
\approx\downarrow & & \downarrow\approx \\
\pi_{19}(S^{13}) & \xrightarrow{\partial_{5_{*19}}} & \pi_{18}(F_5) \\
\parallel & & \\
\mathbb{Z}/2\{\nu_{13}^2\} & &
\end{array}$$

So, $\text{Ker}(\partial_{5_{*19}}) = 0$. Then, we have the exact aequence,

$$0 \to \mathbb{Z}/2\oplus\mathbb{Z}/2 \longrightarrow \pi_{19}(\Sigma^5\mathbb{H}P^2) \longrightarrow 0 \to 0$$

This gives $\pi_{19}(\Sigma^5\mathbb{H}P^2) \approx \mathbb{Z}/2\oplus\mathbb{Z}/2$.

For $k = 4$, we need $\text{cok}(\partial_{4_{*19}})$ and $\text{Ker}(\partial_{4_{*18}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/16\{\sigma_{11}\} \arrow[d, equal] & \mathbb{Z}/8\{\sigma_8\nu_{15}\}\oplus\mathbb{Z}/8\{\nu_8\sigma_{11}\}\oplus\mathbb{Z}/2\{\eta_8\mu_9\} \arrow[d, equal] \\
\pi_{18}(S^{11}) \arrow[r, "\nu_{9*}"] \arrow[d, "\approx"'] & \pi_{18}(S^8) \arrow[d, "onto"] \\
\pi_{19}(S^{12}) \arrow[r, "\partial_{4_{*19}}"] \arrow[d, equal] & \pi_{18}(F_4) \arrow[d, equal] \\
\mathbb{Z}/16\{\sigma_{12}\} & \mathbb{Z}/8\{j_4\nu_8\sigma_{11}\}\oplus\mathbb{Z}/2\{j_4\eta_8\mu_9\}
\end{tikzcd}$$

thus $\mathrm{cok}(\partial_{4_{*19}}) = \mathbb{Z}/2\{j_4\eta_8\mu_9\}$. Next we observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/2\{\nu_{11}^2\} \arrow[d, equal] & \mathbb{Z}/2\{\nu_8^3\}\oplus else \arrow[d, equal] \\
\pi_{17}(S^{11}) \arrow[r, "\nu_{8*}"] \arrow[d, "\approx"'] & \pi_{17}(S^8) \arrow[d, "\approx"] \\
\pi_{18}(S^{12}) \arrow[r, "\partial_{4_{*18}}"] \arrow[d, equal] & \pi_{17}(F_4) \\
\mathbb{Z}/2\{\nu_{12}^2\} &
\end{tikzcd}$$

So, $\mathrm{Ker}(\partial_{4_{*18}}) = 0$. Then, we have the exact aequence,

$$0 \to \mathbb{Z}/2 \longrightarrow \pi_{18}(\Sigma^4\mathbb{H}P^2) \longrightarrow 0 \to 0$$

This gives $\pi_{18}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}/2$.

For $k = 3$, we need $\mathrm{cok}(\partial_{3_{*18}})$ and $\mathrm{Ker}(\partial_{3_{*17}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/16\{\sigma_{10}\} \arrow[d, equal] & \mathbb{Z}/8\{\nu_7\sigma_{10}\}\oplus\mathbb{Z}/2\{\eta_7\mu_8\} \arrow[d, equal] \\
\pi_{17}(S^{10}) \arrow[r, "\nu_{7*}"] \arrow[d, "\approx"'] & \pi_{17}(S^7) \arrow[d, "into"] \\
\pi_{18}(S^{11}) \arrow[r, "\partial_{3_{*18}}"] & \pi_{17}(S^7)\oplus\pi_{17}(S^{17})
\end{tikzcd}$$

thus $\mathrm{cok}(\partial_{3_{*18}}) = \mathbb{Z}/2\{j_3\eta_7\mu_8\}\oplus\mathbb{Z}_{(2)}\{\tilde{b}\}$. Next we observe the commutative diagram,

$$\begin{array}{ccc}
\mathbb{Z}/2\{\nu_{10}^2\} & & \mathbb{Z}/2\{\nu_7^3\} \oplus else \\
\| & & \| \\
\pi_{16}(S^{10}) & \xrightarrow{\nu_{7*}} & \pi_{16}(S^7) \\
\approx \downarrow & & \downarrow \approx \\
\pi_{17}(S^{11}) & \xrightarrow{\partial_{3*17}} & \pi_{16}(F_4) \\
\| & & \\
\mathbb{Z}/2\{\nu_{11}^2\} & &
\end{array}$$

So, $\mathrm{Ker}(\partial_{3_{*17}}) = 0$. Then, we have the exact aequence,

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)} \longrightarrow \pi_{17}(\Sigma^3\mathbb{H}P^2) \longrightarrow 0 \to 0$$

This gives $\pi_{17}(\Sigma^3\mathbb{H}P^2) \approx \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)}$.

For $k = 2$, we need $\mathrm{cok}(\partial_{2_{*17}})$ and $\mathrm{Ker}(\partial_{2_{*16}})$. We observe the commutative diagram,

$$\begin{array}{ccc}
\mathbb{Z}/16\{\sigma_9\} & & \mathbb{Z}/8\{\nu_6\sigma_9\} \oplus \mathbb{Z}/2\{\eta_6\mu_7\} \\
\| & & \| \\
\pi_{16}(S^9) & \xrightarrow{\nu_{6*}} & \pi_{16}(S^6) \\
\approx \downarrow & & \downarrow into\ j_{2*} \\
\pi_{17}(S^{10}) & \xrightarrow{\partial_{2*17}} & \pi_{16}(F_2) \\
\| & & \| \\
\mathbb{Z}/16\{\sigma_{10}\} & & \mathbb{Z}/8\{j_2\nu_6\sigma_9\} \oplus \mathbb{Z}/2\{j_2\eta_6\mu_7\} \oplus \mathbb{Z}/2\{\widehat{j_2}\widehat{\eta_{15}}\}
\end{array}$$

thus $\mathrm{cok}(\partial_{2_{*17}}) = \mathbb{Z}/2\{j_2\eta_6\mu_7\} \oplus \mathbb{Z}/2\{\widehat{j_2}\widehat{\eta_{15}}\}$. Next we observe the commutative diagram,

$$\begin{array}{ccc}
\mathbb{Z}/2\{\nu_9^2\} & & \mathbb{Z}/2\{\nu_6^3\} \oplus else \\
\| & & \| \\
\pi_{15}(S^9) & \xrightarrow{\nu_{6*}} & \pi_{15}(S^6) \\
\approx \downarrow & & \downarrow into\ j_{2*} \\
\pi_{16}(S^{10}) & \xrightarrow{\partial_{2*16}} & \pi_{15}(F_2)
\end{array}$$

So, $\mathrm{Ker}(\partial_{2_{*16}}) = 0$. Then, we have the exact aequence,

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \pi_{16}(\Sigma^2 \mathbb{H}P^2) \longrightarrow 0 \to 0$$

This gives $\pi_{16}(\Sigma^2 \mathbb{H}P^2) \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

For $k = 1$, we need $\mathrm{cok}(\partial_{1_{*16}})$ and $\mathrm{Ker}(\partial_{1_{*15}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}_{(2)}\{\bar{b}\} \oplus \mathbb{Z}/8\{\Sigma\sigma'\} \arrow[d, equal] & \mathbb{Z}/8\{\nu_5\sigma_8\} \oplus \mathbb{Z}/2\{\eta_5\mu_6\} \arrow[d, equal] \\
\pi_{15}(S^8) \arrow[r, "\nu_{5*}"] \arrow[d, "onto"'] & \pi_{15}(S^5) \arrow[d, "into"] \\
\pi_{16}(S^9) \arrow[r, "\partial_{1_{*16}}"] & \pi_{15}(S^5) \oplus \pi_{15}(S^{13})
\end{tikzcd}$$

Here, $\sigma_8 = x_0\bar{b}$ for one $x_0 \in \mathbb{Z}_{(2)}$. By *Page* 67 *and* 70 *of* [1], we have $\nu_5 \circ \Sigma\sigma' = -2y\nu_5\sigma_8$ for one $y \in \mathbb{Z}$. Thus $\mathrm{cok}(\partial_{1_{*16}}) \approx \mathbb{Z}/2\{j_1\eta_5\mu_6\} \oplus \mathbb{Z}/2\{\eta_{13}^2\}$. Next we observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/2\{\nu_8^2\} \arrow[d, equal] & \mathbb{Z}/2\{\nu_5^3\} \oplus else \arrow[d, equal] \\
\pi_{14}(S^8) \arrow[r, "\nu_{5*}"] \arrow[d, "\approx"'] & \pi_{14}(S^5) \arrow[d, "into"] \\
\pi_{15}(S^9) \arrow[r, "\partial_{1_{*15}}"] & \pi_{14}(S^5) \oplus \pi_{14}(S^{13})
\end{tikzcd}$$

So, $\mathrm{Ker}(\partial_{1_{*15}}) = 0$. Then, we have the exact aequence,

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \pi_{15}(\Sigma \mathbb{H}P^2) \longrightarrow 0 \to 0$$

This gives $\pi_{15}(\Sigma \mathbb{H}P^2) \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

For $k = 0$, $\pi_{14}(\mathbb{H}P^2) \approx \pi_{14}(S^{11}) \oplus \pi_{13}(S^3) \approx \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$.

9. $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$

$\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$ is in the stable range $\Leftrightarrow k \geq 15 - 6 = 9$.
For $k \geq 9$, by Lemma 4.1, $\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \approx \mathbb{Z}/128$.
For $k = 7$, we have known that $sk_{24}(F_7) = S^{11}$, $sk_{28}(F_9) = S^{13}$, we observe the

homotopy-commutative diagram with the rows the fibre sequences

,

$$\begin{tikzcd}
F_7 \arrow[r] \arrow[d, "\varphi_{7+2}"] & \Sigma^7\mathbb{H}P^2 \arrow[r] \arrow[d, "\omega_{7+2}"] & S^{15} \arrow[d] \\
\Omega^2 F_9 \arrow[r] & \Omega^2\Sigma^9\mathbb{H}P^2 \arrow[r] & \Omega^2 S^{17}
\end{tikzcd}$$

it induces the commutative diagram with exact rows ,

$$\begin{tikzcd}
 & \mathbb{Z}/8\{j_7\zeta_{11}\} \arrow[d, equal] & & \mathbb{Z}/16\{\sigma_{15}\} \arrow[d, equal] & (\mathbb{Z}/2)^2 \arrow[d, equal] \\
\pi_{23}(S^{15}) \arrow[r, "\partial_{7_{*23}}"] \arrow[d, "\approx"] & \pi_{22}(F_7) \arrow[r, "i_{7*}"] \arrow[d, "\varphi_{7+2_{*22}}\ (\approx)"] & \pi_{22}(\Sigma^7\mathbb{H}P^2) \arrow[r, "p_{7*}"] \arrow[d, "\omega_{7+2_*}"] & \pi_{22}(S^{15}) \arrow[r, "\partial_{7_{*22}}"] \arrow[d, "\approx"] & \pi_{21}(F_7) \arrow[d, "\varphi_{7+2_{*21}}"] \\
\pi_{25}(S^{17}) \arrow[r, "\partial_{9_{*25}}"] & \pi_{24}(F_9) \arrow[r, "i_{9*}"] \arrow[d, equal] & \pi_{24}(\Sigma^9\mathbb{H}P^2) \arrow[r, "p_{9*}"] \arrow[d, equal] & \pi_{24}(S^{17}) \arrow[r, "\partial_{9_{*24}}"] \arrow[d, equal] & \pi_{23}(F_9) \arrow[d, equal] \\
 & \mathbb{Z}/8\{j_9\zeta_{13}\} & \mathbb{Z}/128\{\widehat{a}_9\} & \mathbb{Z}/16\{\sigma_{17}\} & \mathbb{Z}/2
\end{tikzcd}$$

Therefore, i_{9*} is into, and p_{9*} is onto, successively $\partial_{9_{*25}} = 0$, $\partial_{7_{*23}} = 0$, then i_{7_*} is into. For the generators $\nu_{11} \in \pi_{14}(S^{11})$, $\sigma_{14} \in \pi_{20}(S^{14})$, $\Sigma^2\nu_{11} = \nu_{13}, \Sigma^2\sigma_{14} = \sigma_{16}$. Since $\pi_{14}(\Sigma^7\mathbb{H}P^2) \approx \pi_7^s(\mathbb{H}P^2) = 0$, $\nu_{11}\sigma_{14} = 0$ ([1], $Page72$), we have Toda bracket $\{j_7', \nu_{11}, \sigma_{14}\}$. For one $x \in \{j_7', \nu_{11}, \sigma_{14}\} \subseteq \pi_{22}(\Sigma^7\mathbb{H}P^2)$, we have $\Sigma^2 x \in \{j_9', \nu_{13}, \sigma_{16}\} \ni \widehat{a}_9$. By lemma 4.2, we get $ord(\Sigma^2 x) = 128$, hence $ord(x) \geq 128$. And it is clearly that $Im(\partial_{7_{*22}}) \approx \mathbb{Z}/2\ or\ 0$, we assert that $Im(\partial_{7_{*22}}) = 0$. To obtain a contradiction, suppose that $Im(\partial_{7_{*22}}) \approx \mathbb{Z}/2$. So $Ker(\partial_{7_{*22}}) \approx \mathbb{Z}/8$, successively $Im(p_{7_{*22}}) \approx \mathbb{Z}/8$. Then we have the exact sequence,

$$0 \to \mathbb{Z}/8 \to \pi_{22}(\Sigma^7\mathbb{H}P^2) \to \mathbb{Z}/8 \to 0,$$

this gives $\pi_{22}(\Sigma^7\mathbb{H}P^2) \approx \mathbb{Z}/64$, $\mathbb{Z}/8 \oplus \mathbb{Z}/8$, $\mathbb{Z}/16 \oplus \mathbb{Z}/4\ or\ \mathbb{Z}/32 \oplus \mathbb{Z}/2$, it is impossible because $ord(x) \geq 128$. This forces $\mathrm{Im}(\partial_{7_{*22}}) = 0$, successively , p_{7_*} is onto. Then we have the exact sequence,

$$0 \to \mathbb{Z}/8 \to \pi_{22}(\Sigma^7\mathbb{H}P^2) \to \mathbb{Z}/16 \to 0,$$

Since $ord(x) \geq 128$, we get $\pi_{22}(\Sigma^7\mathbb{H}P^2) \approx \mathbb{Z}/128$, and $ord(x) = 128$.

For $k = 8$, we have known that $sk_{26}(F_8) = S^{12}$, we observe the homotopy-commutative diagram with rows fibre sequences ,

$$\begin{tikzcd}
F_8 \arrow[r] \arrow[d, "\varphi_{8+2}"] & \Sigma^8\mathbb{H}P^2 \arrow[r] \arrow[d, "\omega_{8+2}"] & S^{16} \arrow[d] \\
\Omega^2 F_9 \arrow[r] & \Omega^2\Sigma^9\mathbb{H}P^2 \arrow[r] & \Omega^2 S^{17}
\end{tikzcd}$$

it induces the commutative diagram with exact rows ,

$$\begin{tikzcd}
 & \mathbb{Z}/8\{j_8\zeta_{12}\}\oplus\mathbb{Z}_{(2)}\{\widehat{c}\} \arrow[d, equal] & & \mathbb{Z}/16\{\sigma_{16}\} \arrow[d, equal] & \mathbb{Z}/2 \arrow[d, equal] \\
\pi_{24}(S^{16}) \arrow[r, "\partial_{8_{*24}}"] \arrow[d, "\approx"] & \pi_{23}(F_8) \arrow[r, "i_{8*}"] \arrow[d, "\varphi_{8+1_{*23}}"] & \pi_{23}(\Sigma^8\mathbb{H}P^2) \arrow[r, "p_{8*}"] \arrow[d, "\omega_{8+1_*}"] & \pi_{23}(S^{16}) \arrow[r, "\partial_{8_{*23}}"] \arrow[d, "\approx"] & \pi_{22}(F_8) \arrow[d, "\varphi_{8+1_{*22}}\ (\approx)"] \\
\pi_{25}(S^{17}) \arrow[r, "0"] \arrow[d, equal] & \pi_{24}(F_9) \arrow[r, "i_{9*}"] \arrow[d, equal] & \pi_{24}(\Sigma^9\mathbb{H}P^2) \arrow[r, "p_{9*}"] \arrow[d, equal] & \pi_{24}(S^{17}) \arrow[r, "0"] \arrow[d, equal] & \pi_{23}(F_9) \arrow[d, equal] \\
\mathbb{Z}/2\oplus\mathbb{Z}/2 & \mathbb{Z}/8\{j_9\zeta_{13}\} & \mathbb{Z}/128\{\widehat{a_9}\} & \mathbb{Z}/16\{\sigma_{17}\} & \mathbb{Z}/2
\end{tikzcd}$$

here, $\varphi_{8+1_{*23}}$ satisfies $\varphi_{8+1_{*23}}(j_8\zeta_{12}) = j_9\zeta_{13}$. Therefore, $\partial_{8_{*24}} = 0$, so i_{8_*} is into, and obviously p_{8_*} is onto. Then we have the exact sequence,

$$0 \to \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2^3 \to \pi_{22}(\Sigma^7\mathbb{H}P^2) \to \mathbb{Z}/2^4 \to 0.$$

By Lemma 2.2.2, we have $\pi_{22}(\Sigma^7\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus B$, where $B = \mathbb{Z}/2^r \oplus \mathbb{Z}/2^s, \mathbb{Z}/2^7$ *or* $\mathbb{Z}/2^3 \oplus \mathbb{Z}/2^4$ $(3 \le r \le 6,\ 0 \le s \le 3)$. We observe the element $x \in \pi_{22}(\Sigma^7\mathbb{H}P^2)$ in the proof of the case $k = 7$,we have known $ord(x) = ord(\Sigma^2 x) = 2^7$, then we get $ord(\Sigma x) = 2^7$, that is, $\pi_{24}(\Sigma^9\mathbb{H}P^2)$ has an element of order 2^7. Hence, $\pi_{24}(\Sigma^9\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2^7$.

For $k = 6$, we have known that $sk_{22}(F_6) = S^{10}$, we observe the homotopy-commutative diagram

$$\begin{tikzcd}
S^{13} \arrow[r, "\nu_{10}"] \arrow[d, "\subseteq"'] & S^{10} \arrow[d, "\subseteq"] \\
\Omega S^{14} \arrow[r, "\partial_6"] & F_6
\end{tikzcd}$$

And it induces the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/2\{\varepsilon_{13}\}\oplus\mathbb{Z}/2\{\bar{\nu}_{13}\} \arrow[d, equal] & \mathbb{Z}/8\{\zeta_{10}\} \arrow[d, equal] \\
\pi_{21}(S^{13}) \arrow[r, "\nu_{10*}"] \arrow[d, "\approx"'] & \pi_{21}(S^{10}) \arrow[d, "\approx"] \\
\pi_{22}(S^{14}) \arrow[r, "\partial_{6_{*22}}"] \arrow[d, equal] & \pi_{21}(F_6) \arrow[d, equal] \\
\mathbb{Z}/2\{\varepsilon_{14}\}\oplus\mathbb{Z}/2\{\bar{\nu}_{14}\} & \mathbb{Z}/8\{j_6\zeta_{10}\}
\end{tikzcd}$$

By *Page* 70 *of* [1], we have $\nu_6\bar{\nu}_9 = \nu_6\varepsilon_9 = 2\bar{\nu}_6\nu_{14}$, hence $\nu_8\bar{\nu}_{11} = \nu_8\varepsilon_{11} = 2\bar{\nu}_8\nu_{16} = 0$, therefore, $\nu_{10_*} : \pi_{21}(S^{13}) \to \pi_{21}(S^{10})$ is trivial, successively, $\partial_{6_{*22}} = 0$, and $\mathrm{cok}(\partial_{6_{*22}}) \approx \pi_{21}(F_6) = \mathbb{Z}/8\{j_6\zeta_{10}\}$.

Then we observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/16\{\sigma_{13}\} \arrow[d, equal] & \mathbb{Z}/4\{\sigma_{10}\nu_{17}\}\oplus\mathbb{Z}/2\{\eta_{10}\mu_{11}\} \arrow[d, equal] \\
\pi_{20}(S^{13}) \arrow[r, "\nu_{10*}"] \arrow[d, "\approx"'] & \pi_{20}(S^{10}) \arrow[d, "\approx"] \\
\pi_{21}(S^{14}) \arrow[r, "\partial_{6_{*21}}"] \arrow[d, equal] & \pi_{20}(F_6) \arrow[d, equal] \\
\mathbb{Z}/16\{\sigma_{14}\} & \mathbb{Z}/4\{j_6\sigma_{10}\nu_{17}\}\oplus\mathbb{Z}/2\{j_6\eta_{10}\mu_{11}\}
\end{tikzcd}$$

By *Page* 72 *of* [1], we have $\nu_{10}\sigma_{13} = 2t\sigma_{10}\nu_{17}$ for an odd t, this gives $\mathrm{Ker}(\partial_{6_{*21}}) = \mathbb{Z}/8\{2\sigma_{14}\}$.

So we get the exact sequence,

$$0 \to \mathbb{Z}/8\{j_6\zeta_{10}\} \to \pi_{21}(\Sigma^6\mathbb{H}P^2) \to \mathbb{Z}/8\{2\sigma_{14}\} \to 0.$$

So $\pi_{21}(\Sigma^6\mathbb{H}P^2) \approx \mathbb{Z}/64, \mathbb{Z}/8\oplus\mathbb{Z}/8, \mathbb{Z}/16\oplus\mathbb{Z}/4$ or $\mathbb{Z}/32\oplus\mathbb{Z}/2$.

Now we consider $p_6\circ\{j'_6, \nu_{10}, 2\sigma_{13}\} = -\{p_6, j'_6, \nu_{10}\}\circ 2\sigma_{14} \ni -2\sigma_{14}$, so $\exists y \in \{j'_6, \nu_{10}, 2\sigma_{13}\}$ such that $p_{6*}(y) = -2\sigma_{14}$.

While $8y \in 8\{j'_6, \nu_{10}, 2\sigma_{13}\} = -j'_6\{\nu_{10}, 2\sigma_{13}, 8\iota_{20}\} \ni \pm j'_6\zeta_{10}$, mod 0 . Then $8y = \pm j'_6\zeta_{10}$. Because $j'_{6*_{20}}$ is into, thus $ord(j'_6\zeta_{10}) = 8$, then $ord(8y) = 8$, hence, $ord(y) = 64$. Therefore, $\pi_{21}(\Sigma^6\mathbb{H}P^2) \approx \mathbb{Z}/64$.

For $k = 5$, by lemma 2.9.5, $sk_{32}(F_5) = S^9 \cup_{f_5} e^{21}$, $f_5 = \bar{\nu}_9\nu_{17}$. Because $\pi_{20}(S^9) = \mathbb{Z}/8\{\zeta_9\} \oplus \mathbb{Z}/2\{\bar{\nu}_9\nu_{17}\}$, Hence $j_{5_*}(\bar{\nu}_9\nu_{17}) = 0$, $\pi_{20}(F_5) = \mathbb{Z}/8\{j_5\zeta_9\}$. Now, we observe the commutative diagram ,

$$\begin{tikzcd}
\pi_{20}(S^{12}) \arrow[r, "\nu_{9_*}"] \arrow[d, "\approx"'] & \pi_{20}(S^9) \arrow[d, "onto"', "j_{5_*}"] \\
\pi_{21}(S^{13}) \arrow[r, "\partial_{5_{*21}}"] & \pi_{20}(F_5)
\end{tikzcd}$$

By $Page$ 70 of [1], we have $\nu_6\bar{\nu}_9 = \nu_6\varepsilon_9 = 2\bar{\nu}_6\nu_{14}$, hence $\nu_9\bar{\nu}_{12} = \nu_9\varepsilon_{12} = 2\bar{\nu}_9\nu_{17} = 0$, therefore, $\nu_{9_*} : \pi_{20}(S^{12}) \to \pi_{20}(S^9)$ is trivial, successively $\partial_{5_{*21}} = 0$, $\text{cok}(\partial_{5_{*21}}) \approx \pi_{20}(F_5) = \mathbb{Z}/8\{j_5\zeta_8\}$. Next, we observe the commutative diagram ,

$$\begin{tikzcd}
 & \mathbb{Z}/8\{\sigma_9\nu_{16}\} \oplus \mathbb{Z}/2\{\eta_9\mu_{10}\} \arrow[d, equal] \\
\pi_{19}(S^{12}) \arrow[r, "\nu_{9_*}"] \arrow[d, "\approx"'] & \pi_{19}(S^9) \arrow[d, "\approx"] \\
\pi_{20}(S^{13}) \arrow[r, "\partial_{5_{*20}}"] & \pi_{19}(F_5)
\end{tikzcd}$$

By $Page$ 72 of [1], we have $\nu_9\sigma_{12} = 2t\sigma_9\nu_{16}$ for one odd t, so $\text{Ker}(\partial_{5_{*20}}) \approx \text{Ker}(\nu_{9_*}) \approx \mathbb{Z}/4$, successively $\text{Ker}(\partial_{5_{*20}}) = \mathbb{Z}/4\{4\sigma_{13}\}$. Thus we have the exact sequence,

$$0 \to \mathbb{Z}/8\{j_5\zeta_8\} \longrightarrow \pi_{20}(\Sigma^5\mathbb{H}P^2) \longrightarrow \mathbb{Z}/4\{4\sigma_{13}\} \to 0,$$

This gives $\pi_{20}(\Sigma^5\mathbb{H}P^2) \approx \mathbb{Z}/8 \oplus \mathbb{Z}/4, \mathbb{Z}/16 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/32$.
Comparing with the sequence of $\pi_{21}(\Sigma^6\mathbb{H}P^2)$ we observe the commutative diagram with short exact rows,

$$\begin{tikzcd}
\mathbb{Z}/8 \arrow[r] \arrow[d, "\approx"'] & \pi_{20}(\Sigma^5\mathbb{H}P^2) \arrow[r] \arrow[d, "w_{5+1_*}"'] & \mathbb{Z}/4 \arrow[d, "into"] \\
\mathbb{Z}/8 \arrow[r] & \mathbb{Z}/64 \arrow[r] & \mathbb{Z}/8
\end{tikzcd}$$

By the snake lemma, we have w_{5+1_*} is into. So $\pi_{20}(\Sigma^5\mathbb{H}P^2) \approx \mathbb{Z}/32$.

For $k = 3$, notice that we knew that $sk_{20}(F_3) = sk_{20}(\,J(M_{S^7}, S^{10})\,) = S^{17} \vee S^{17}$, and we have the exact sequence

$$\pi_{19}(S^{11}) \xrightarrow{\partial_{3*19}} \pi_{18}(F_3) \xrightarrow{i_{3*}} \pi_{18}(\Sigma^3\mathbb{H}P^2) \xrightarrow{p_{3*}} \pi_{18}(S^{11}) \xrightarrow{\partial_{3*18}} \pi_{17}(F_3)$$

induces the exact sequence

$$0 \longrightarrow cok(\partial_{3_{*19}}) \xrightarrow{\bar{i}_{3*}} \pi_{18}(\Sigma^3\mathbb{H}P^2) \xrightarrow{\bar{p}_{3*}} Ker(\partial_{3_{*18}}) \longrightarrow 0.$$

We need to know $\mathrm{cok}(\partial_{3_{*19}})$ and $\mathrm{Ker}(\partial_{3_{*18}})$.

Firstly, we observe the homotopy-commutative diagram ,

$$\begin{tikzcd}
J(S^{10}, S^{10}) \arrow[d, equal] & J(M_{S^7}, S^{10}) \arrow[d, equal] \\
\Omega S^{11} \arrow[r, "\partial_3"] \arrow[d, "H_2'"'] & F_3 \arrow[d, "H_2"] \\
\Omega S^{21} \arrow[r] & \Omega S^{18}
\end{tikzcd}$$

Here, we identify $J(M_{S^7}, S^{10})$ with F_3, and identify $J(S^{10}, S^{10}) = J(S^{10})$ with ΩS^{11}, and H_2, H_2' are the second relative James-Hopf invariants which were introduced in 2.8. This diagram induces commutative diagram of homotopy groups,

$$\begin{tikzcd}
\pi_{19}(S^{11}) \arrow[r, "\partial_{3_*}"] \arrow[d, "H'_{2_*}"'] & \pi_{18}(J(M_{S^7}, S^{10})) \arrow[d, "H_{2*}"] \\
\pi_{19}(S^{21}) \arrow[r] & \pi_{19}(S^{18})
\end{tikzcd}$$

This gives $H_{2_*}\partial_{3_*} = 0$.

Next, we observe the homotopy-commutative diagram,

$$\begin{tikzcd}
S^{17} \arrow[r, "\subseteq"'] \arrow[dr, "\Omega\Sigma"'] & J(M_{S^7}, S^{10}) \arrow[r, "pinch\ S^7\ to\ *", "q_1"'] \arrow[d, "H_2"] & S^{17} \cup e^{27} \cup \cdots \arrow[dl, "\widehat{H_2}"'] \\
& \Omega S^{18} & S^{17} \arrow[l, "\Omega\Sigma"] \arrow[u, "j_1"]
\end{tikzcd}$$

Here, $\widehat{H_2}$ is given by $q(x) \mapsto H_2(x), \forall x \in J(M_{S^7}, S^{10})$ and $\widehat{H_2}$ is clearly well-defined since $sk_7(\Omega S^{18}) = *$ and we could regard H_2 as a celluar map.

This diagram induces commutative diagram of homotopy groups,

$$\begin{tikzcd}
\pi_{18}(S^{17}) \arrow[r] \arrow[dr, "\Sigma"'] & \pi_{18}(S^{7}) \oplus \pi_{18}(S^{17}) \arrow[r, "q_{1_*}"] \arrow[d, "H_{2_*}"] & \pi_{18}(S^{17}) \arrow[dl, "\widehat{H}_{2_*}"] \\
& \pi_{19}(S^{18}) & \pi_{18}(S^{17}) \arrow[l, "\Sigma"] \arrow[u, "id"']
\end{tikzcd}$$

So $H_{2_*} = \widehat{H}_{2_*} q_{1_*} = \Sigma \circ q_{1_*}$. Since $H_{2_*}\partial_{3_*} = 0$, we have $\Sigma \circ q_{1_*}\partial_{3_*} = 0$, obviously Σ is an isomorphism, successively $q_{1_*}\partial_{3_*} = 0$, here, we used the identifications $\pi_{18}(J(M_{S^7}, S^{10})) = \pi_{18}(S^7) \oplus \pi_{18}(S^{17})$ and $\pi_{18}(S^{17} \cup e^{27} \cup \cdots) = \pi_{18}(S^{17})$. Hence $q_{1_*}\partial_{3_*} = 0$, this means for the composition,

$$\begin{array}{ccccc}
\pi_{19}(S^{11}) & \xrightarrow{\partial_{3_{*19}}} & \pi_{18}(S^{7}) \oplus \pi_{18}(S^{17}) & \xrightarrow{q_{1_*}} & \pi_{18}(S^{17}) \\
x & \mapsto & (x_1, x_2) & \mapsto & x_2
\end{array}$$

we have $\partial_{3_{*19}}(x) = (x_1, 0)$.

Then, we observe the homotopy-commutative diagram ,

$$\begin{tikzcd}
\Omega S^{11} \arrow[r, "\partial_3"] \arrow[d, "id"'] & F_3 \arrow[d, "\phi_3"] \\
\Omega S^{11} \arrow[r, "\Omega\nu_8"] & \Omega S^8
\end{tikzcd}$$

It induces the commutative diagram ,

$$\begin{tikzcd}
\mathbb{Z}/2\{\overline{\nu}_{11}\} \oplus z/2\{\varepsilon_{11}\} \arrow[d, equal] & \\
\pi_{19}(S^{11}) \arrow[r, "\partial_{3_{*19}}"] \arrow[d, "id"'] & \pi_{18}(S^{7}) \oplus \pi_{18}(S^{17}) \arrow[d, "\phi_{3_*}"] \\
\pi_{19}(S^{11}) \arrow[r, "\nu_{8_*}"] & \pi_{19}(S^{8}) \arrow[d, equal] \\
& \mathbb{Z}/8\{\zeta_8\} \oplus \mathbb{Z}/2\{\overline{\nu}_8\nu_{16}\}
\end{tikzcd}$$

We assert that $\phi_{3_*}|_{\pi_{18}(S^7)}$ is the suspension homomorphism. By Remark 2.7.2, there exists homotopy-commutative diagram,

$$\begin{tikzcd}
S^7 \arrow[r, "j_3", "\subseteq"'] \arrow[d, "\Omega\Sigma"'] & F_3 \arrow[r, "i_3"] \arrow[d, "\phi_3"'] & \Sigma^3\mathbb{H}P^2 \arrow[r, "p_3"] \arrow[d, "\mathcal{B}_3"'] & S^{11} \arrow[d, "id"'] & \\
\Omega S^8 \arrow[r, "id"] & \Omega S^8 \arrow[r, "d_3"] & \mathcal{G}_3 \arrow[r, "\bar{j}_3"] & S^{11} \arrow[r, "\nu_8"] & S^8
\end{tikzcd}$$

It induces the commutative diagram,

$$\begin{tikzcd}
\pi_{18}(S^7) \arrow[r, "j_{3_*}"] \arrow[d, "\Sigma"'] & \pi_{18}(S^7) \oplus \pi_{18}(S^{17}) \arrow[d, "\phi_{3_*}"'] \\
\pi_{19}(S^8) \arrow[r, "id"] & \pi_{19}(S^8)
\end{tikzcd}$$

Hence , $\phi_{3_*}|_{\pi_{18}(S^7)} = \phi_{3_*} j_{3_*} = \Sigma : \pi_{18}(S^7) \to \pi_{19}(S^8)$, and it is an isomorphism by $Page$ 66 of

[1], $Toda$.

Next, by $Page$ 70 of [1], $Toda$, there are formulas $\nu_6\bar{\nu}_9 = \nu_6\varepsilon_9$, $\nu_6\varepsilon_9 = 2\bar{\nu}_6\nu_{14}$, successively, their images of Σ^2 satisfying $\nu_8\bar{\nu}_{11} = \nu_8\varepsilon_{11} = 2\bar{\nu}_8\nu_{16} = 0$. So, $\nu_{8_*} = 0$. Then $\phi_{3_*}\partial_{3_{*19}} = \phi_{3_*}|_{\pi_{18}(S^7)} \circ \partial_{3_{*19}} = \Sigma \circ \partial_{3_{*19}} = 0$, while $\Sigma : \pi_{18}(S^7) \to \pi_{19}(S^8)$ is an isomorphism, so $\partial_{3_{*19}} = 0$.

Then we observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/8\{\nu_7\sigma_{10}\} \oplus \mathbb{Z}/2\{\eta_7\mu_8\} \arrow[d, equal] & \\
\pi_{17}(S^7) \arrow[r, "j_{3_*}"] \arrow[d, "\Sigma"'] & \pi_{17}(S^7) \oplus \pi_{17}(S^{17}) \arrow[d, "\phi_{3_*}"'] \\
\pi_{18}(S^8) \arrow[r, "id"] \arrow[d, equal] & \pi_{18}(S^8) \\
\mathbb{Z}/8\{\nu_8\sigma_{11}\} \oplus \mathbb{Z}/2\{\eta_8\mu_9\} \oplus \mathbb{Z}/8\{\sigma_8\nu_{15}\} &
\end{tikzcd}$$

Hence , $\phi_{3_*}|_{\pi_{17}(S^7)} = \phi_{3_*} j_{3_*} = \Sigma : \pi_{17}(S^7) \to \pi_{18}(S^8)$, and it is into given by $\nu_7\sigma_{10} \mapsto \nu_8\sigma_{11}$, $\eta_7\mu_8 \mapsto \eta_8\nu_9$ according to $Page$ 66 of [1], $Toda$.

Now, observe the commutative diagram,

$$\begin{tikzcd}
\pi_{18}(S^{11}) \arrow[r, "\partial_{3_{*18}}"] \arrow[d, "id"'] & \pi_{17}(S^7) \oplus \pi_{17}(S^{17}) \arrow[d, "\phi_{3_*}"] \\
\pi_{18}(S^{11}) \arrow[r, "\nu_{8_*}"] \arrow[d, equal] & \pi_{18}(S^8) \arrow[d, equal] \\
\mathbb{Z}/16\{\sigma_{11}\} & \mathbb{Z}/8\{\nu_8\sigma_{11}\} \oplus \mathbb{Z}/2\{\eta_8\mu_9\} \oplus \mathbb{Z}/8\{\sigma_8\nu_{15}\}
\end{tikzcd}$$

So, $\mathrm{Im}(\nu_{8_*}) = \mathbb{Z}/8\{\nu_8\sigma_{11}\}$. Since $\phi_{3_*}\partial_{3_{*18}} = \phi_{3_*}|_{\pi_{17}(S^7)} \circ \partial_{3_{*18}} = \nu_{8_*}$, and $\phi_{3_*}|_{\pi_{17}(S^7)}$ is into. We get $\mathrm{Ker}(\partial_{3_{*18}})$=$\mathrm{Ker}(\nu_{8_*}) = \mathbb{Z}/2\{8\sigma_{11}\}$. Recall that we have $\partial_{3_{*19}} = 0$, thus, we have the exact sequence,

$$\begin{array}{c} \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2 \\ \| \\ 0 \to \pi_{18}(S^7) \oplus \pi_{18}(S^{17}) \longrightarrow \pi_{18}(\Sigma^3\mathbb{H}P^2) \longrightarrow \mathbb{Z}/2\{8\sigma_{11}\} \to 0. \end{array}$$

It gives $\pi_{18}(\Sigma^3\mathbb{H}P^2) \approx \mathbb{Z}/16 \oplus (\mathbb{Z}/2)^2,\ \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2\ or\ \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^3$.
Similarly as above, comparing with the short exact sequence of $\pi_{20}(\Sigma^5\mathbb{H}P^2) \approx \mathbb{Z}/32$, by the Snake Lemma, we have $\mathrm{cok}(w_{3+2_*}) \approx \mathbb{Z}/2$. This suggests that $\exists u \in \pi_{18}(\Sigma^3\mathbb{H}P^2)$ such that $ord(w_{3+2_*}(u)) = 16$. Thus $ord(u) \geq 16$, so $\pi_{18}(\Sigma^3\mathbb{H}P^2) \approx \mathbb{Z}/16 \oplus (\mathbb{Z}/2)^2$.

For $k = 4$, we need $\mathrm{cok}(\partial_{4_{*20}})$ and $\mathrm{Ker}(\partial_{4_{*19}})$. We observe the commutative diagram,

$$\begin{tikzcd} \mathbb{Z}/2\{\bar{\nu}_{11}\} \oplus \mathbb{Z}/2\{\varepsilon_{11}\} \arrow[d, equal] & \mathbb{Z}/8\{\zeta_8\} \oplus \mathbb{Z}/2\{\bar{\nu}_8\nu_{16}\} \arrow[d, equal] \\ \pi_{19}(S^{11}) \arrow[r, "\nu_{8_*}"] \arrow[d, "\approx"'] & \pi_{19}(S^8) \arrow[d, "into"] \\ \pi_{20}(S^{12}) \arrow[r, "\partial_{4_{*20}}"] & \pi_{19}(F_4) \end{tikzcd}$$

By *Page* 70 *of* [1], we have $\nu_8\bar{\nu}_{11} = \nu_8\varepsilon_{11} = 0$, so $\nu_{8_*} = 0$, thus $\mathrm{cok}(\partial_{4_{*20}}) \approx \pi_{19}(F_4) = \mathbb{Z}/8\{j_4\zeta_8\} \oplus \mathbb{Z}/2\{j_4\bar{\nu}_8\nu_{16}\} \oplus \mathbb{Z}_{(2)}\{\acute{a}\}$. Next we observe the commutative diagram,

$$\begin{tikzcd} \mathbb{Z}/16\{\sigma_{12}\} \arrow[d, equal] & \mathbb{Z}/8\{j_4\sigma_8\nu_{15}\} \oplus \mathbb{Z}/2\{j_4\eta_8\mu_9\} \arrow[d, equal] \\ \pi_{19}(S^{12}) \arrow[r, "\partial_{4_{*19}}"] \arrow[d, "id"'] & \pi_{18}(F_4) \arrow[d, "\approx"] \\ \pi_{19}(S^{12}) \arrow[r, "\nu_{9_*}"] & \pi_{19}(S^9) \end{tikzcd}$$

By *Page* 72 *of* [1], we have $\nu_9\sigma_{12} = 2t\sigma_9\nu_{16}$ for one odd t. So, $\mathrm{Ker}(\partial_{4_{*19}}) = \mathbb{Z}/4\{4\sigma_{12}\}$. Then, we have the exact aequence,

$$0 \to \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)} \longrightarrow \pi_{19}(\Sigma^4\mathbb{H}P^2) \longrightarrow \mathbb{Z}/4 \to 0$$

By lemma 2.2.1, we have $\pi_{19}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus K$, where $K = \mathbb{Z}/8\oplus\mathbb{Z}/2, \mathbb{Z}/8\oplus\mathbb{Z}/4, \mathbb{Z}/16\oplus \mathbb{Z}/2, \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2, \mathbb{Z}/32 \oplus \mathbb{Z}/2,\ (\mathbb{Z}/8)^2\ or\ \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$.

We noticed that $p_4\circ\{j'_4, \nu_8, 8\sigma_{11}\} = -\{p_4, j'_4, \nu_8\}\circ 8\sigma_{12} \ni -8\sigma_{12}$, so $\exists z \in \{j'_4, \nu_8, 8\sigma_{11}\}$, such that $p_{4_*}(z) = -8\sigma_{12}$. While $8z \in -j'_4\{\nu_8, 8\sigma_{11}, 8\iota_{18}\} \ni \pm 4j'_4\zeta_8$, mod 0. So $ord(8z) = 2$, then $ord(z) = 16$. Then ,$\pi_{19}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus \mathbb{Z}/32 \oplus \mathbb{Z}/2$ or $\mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/2$.
To exclude to the extra solution of these two , we consider the fibre sequence ($\mathcal{U}_4 = J(M_{S^{11}}, \Sigma^3\mathbb{H}P^2)$ is the fibre of $j'_4 : S^8 \hookrightarrow \Sigma^4\mathbb{H}P^2$)

$$\mathcal{U}_4 \longrightarrow S^8 \xrightarrow{j'_4} \Sigma^4\mathbb{H}P^2$$

By checking homology , and by $\Sigma\mathcal{U}_4$ is splitting and $\Sigma : \pi_{17}(S^{11}) \to \pi_{18}(S^{12})$ is an isomorphism , we get $\mathrm{sk}_{21}(\mathcal{U}_4) = S^{11} \vee S^{18}$. So there's a fibre sequence up to dimension 20 (here, $\mathbf{a} = \nu_8 \vee f$ for some f could be got by Lemma 2.7.9),

$$S^{11} \vee S^{18} \xrightarrow{\mathbf{a} = \nu_8\vee f} S^8 \xrightarrow{j'_4} \Sigma^4\mathbb{H}P^2$$

it induces the exact sequence,

$$\pi_{19}(S^8) \xrightarrow{j'_{4_*}} \pi_{19}(\Sigma^4\mathbb{H}P^2) \longrightarrow \pi_{18}(S^{11}) \oplus \pi_{18}(S^{18}) \xrightarrow[(\nu_{8_*}, f_*)]{\mathbf{a}_*} \pi_{18}(S^8)$$

By the result of $\mathrm{cok}(\partial_{4_{*20}})$, we know that j'_{4_*} is into, so $\mathrm{Im}(j'_{4_*}) \approx \pi_{19}(S^8) \approx \mathbb{Z}/8 \oplus \mathbb{Z}/2$.
We know $\nu_{8_*} : \mathbb{Z}/16\{\sigma_{11}\} \to \mathbb{Z}/8\{\nu_8\sigma_{11}\} \oplus else$ has image $\mathbb{Z}/8\{\nu_8\sigma_{11}\}$, so $\mathrm{Ker}(\mathbf{a}_*) \approx \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)}$. So we have the exact sequence,

$$0 \longrightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/2 \longrightarrow \pi_{19}(\Sigma^4\mathbb{H}P^2) \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)} \longrightarrow 0$$

This gives $\pi_{19}(\Sigma^4\mathbb{H}P^2)$ cannot have elements of order 32. While we've already known $\pi_{19}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus \mathbb{Z}/32 \oplus \mathbb{Z}/2$ or $\mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/2$.
Hence, $\pi_{19}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/2$.

For $k = 2$, we have the exact sequence

$$\pi_{18}(S^{10}) \xrightarrow{\partial_{2_{*18}}} \pi_{17}(F_2) \xrightarrow{i_{2*}} \pi_{17}(\Sigma^2\mathbb{H}P^2) \xrightarrow{p_{2*}} \pi_{17}(S^{10}) \xrightarrow{\partial_{2_{*17}}} \pi_{16}(F_2)$$

We need $\mathrm{cok}(\partial_{2_{*18}})$ and $\mathrm{Ker}(\partial_{2_{*17}})$. Firstly, we observe the homotopy-commutative diagram, where H_2' and H_2 are the 2nd relative James-Hopf invariants ,

$$\begin{tikzcd}
J(S^9) \arrow[r, "\partial_2"] \arrow[dd, "H_2'"] & J(M_{S^6}, S^9) \arrow[dd, "H_2"] & J_2(M_{S^6}, S^9) \arrow[l, "\supseteq"'] \arrow[r, "pinch\ M_{S^6}"] \arrow[d, "\Omega\Sigma"'] & S^{15} \arrow[d, "\Omega\Sigma"] \\
 & & \Omega(S^7 \vee S^{16}) \arrow[r] \arrow[d, "\Omega\, pinch"] & \Omega S^{16} \arrow[d, "id"] \\
J(S^{18}) \arrow[r] & J(S^{15}) & \Omega S^{16} \arrow[l, "\simeq"'] \arrow[r, "id"] & \Omega S^{16}
\end{tikzcd}$$

it induces the commutative diagram (after several identifications with the isomorphism groups),

$$\begin{tikzcd}
\pi_{18}(S^{10}) \arrow[r, "\partial_{2*18}"] \arrow[d, "H'_{2_*}=0"] & \pi_{17}(F_2) \arrow[r, "\breve{p}'_{2_*}"] \arrow[d, "H_{2_*}"] & \pi_{17}(S^{15}) \arrow[d, "\approx"] \\
\pi_{18}(S^{19}) \arrow[r] & \pi_{18}(S^{16}) \arrow[r, "id"] & \pi_{18}(S^{16})
\end{tikzcd}$$

Thus, $H_{2_*} \circ \partial_{2_{*18}} = 0$, this means the following the composition is zero,

$$\pi_{18}(S^{10}) \xrightarrow{\partial_{2*18}} \pi_{17}(F_2) \xrightarrow{proj.} \pi_{17}(S^{15}) \xrightarrow{\approx} \pi_{18}(S^{16})$$

So, for any $u \in \pi_{18}(S^{10})$, $\partial_{2_{*18}}(u) = x_1 j_2\zeta_6 + x_2 j_2\bar{\nu}_6\nu_{14} + x_3\widehat{j_2\eta^2_{15}}$ implies $x_3 \equiv 0\ (mod\ 2)$. Hence, $\mathrm{Im}(\partial_{2_{*18}}) \subseteq \mathbb{Z}/8\{j_2\zeta_6\} \oplus \mathbb{Z}/2\{j_2\bar{\nu}_6\nu_{14}\}$. Then we consider the homotopy-commutative diagram,

$$\begin{tikzcd}
\Omega S^{10} \arrow[r, "\partial_2"] \arrow[d, "id"] & F_2 \arrow[d, "\phi_2"] & S^6 \arrow[l, "j_2"'] \arrow[dl, "\Omega\Sigma"] \\
\Omega S^{10} \arrow[r, "\Omega\nu_7"] & \Omega S^7 &
\end{tikzcd}$$

it induces the commutative diagram (here $p : \pi_{17}(F_2) \longrightarrow \pi_{17}(F_2)/\langle\widehat{j_2\eta^2_{15}}\rangle$ is the projection),

$$\begin{tikzcd}
\pi_{18}(S^{10}) \arrow[r, "p\circ\partial_{2*18}"] \arrow[d, "id"] & \pi_{17}(F_2)/\langle\widehat{j_2\eta_{15}^2}\rangle \arrow[d, "\underline{\phi}_{2*}"'] & \pi_{17}(S^6) \arrow[l, "\underline{j}_{2*}"'] \arrow[dl, "\Sigma"] \\
\pi_{18}(S^{10}) \arrow[r, "\nu_{7*}"] & \pi_{18}(S^7) &
\end{tikzcd}$$

Since $\pi_{18}(S^{10}) = \mathbb{Z}/2\{\bar{\nu}_{10}\}\oplus\mathbb{Z}/2\{\varepsilon_{10}\}$, $\pi_{17}(F_2)/\langle\widehat{j_2\eta_{15}^2}\rangle = \mathbb{Z}/8\{j_2\zeta_6\}\oplus\mathbb{Z}/2\{j_2\bar{\nu}_6\nu_{14}\}$, $\pi_{17}(S^6) = \mathbb{Z}/8\{\zeta_6\}\oplus\mathbb{Z}/4\{\bar{\nu}_6\nu_{14}\}$, $\pi_{18}(S^7) = \mathbb{Z}/8\{\zeta_7\}\oplus\mathbb{Z}/2\{\bar{\nu}_7\nu_{15}\}$, so $\underline{\phi}'_{2_*}$ is an isomorphism. According to *Page* 70 *of* [1], we have $\nu_{7_*} = 0$. Thus, $p\circ\partial_{2_{*18}} = 0$, successively, $\partial_{2_{*18}} = 0$ $\mathrm{cok}(\partial_{2_{*18}}) = \pi_{17}(F_2) = \mathbb{Z}/8\{j_2\zeta_6\}\oplus\mathbb{Z}/2\{j_2\bar{\nu}_6\nu_{14}\}\oplus\mathbb{Z}/2\{\widehat{j_2\eta_{15}^2}\}$.

Next, we observe the commutative diagram ,

$$\begin{tikzcd}
 & \mathbb{Z}/8\{\nu_6\sigma_9\}\oplus\mathbb{Z}/2\{\eta_6\mu_7\} \arrow[d, equal] \\
\pi_{16}(S^9) \arrow[r, "\nu_{6*}"] \arrow[d, "\approx"'] & \pi_{16}(S^6) \arrow[d, "into"] \\
\pi_{17}(S^{10}) \arrow[r, "\partial_{2*17}"] \arrow[d, equal] & \pi_{16}(F_2) \arrow[d, equal] \\
\mathbb{Z}/16\{\sigma_{10}\} & \mathbb{Z}/8\{j_2\nu_6\sigma_9\}\oplus\mathbb{Z}/2\{j_2\eta_6\mu_7\}\oplus\mathbb{Z}/2\{\widehat{j_2\eta_{15}}\}
\end{tikzcd}$$

Hence, $\mathrm{Ker}(\partial_{2_{*17}}) = \mathbb{Z}/2\{8\sigma_{10}\}$. Therefore, we get the exact sequence,

$$0\to\mathbb{Z}/8\{j_2\zeta_6\}\oplus\mathbb{Z}/2\{j_2\bar{\nu}_6\nu_{14}\}\oplus\mathbb{Z}/2\{\widehat{j_2\eta_{15}^2}\}\longrightarrow\pi_{17}(\Sigma^2\mathbb{H}P^2)\longrightarrow\mathbb{Z}/2\{8\sigma_{10}\}\to 0$$

This gives $\pi_{17}(\Sigma^2\mathbb{H}P^2)\approx\mathbb{Z}/8\oplus(\mathbb{Z}/2)^3$, $\mathbb{Z}/16\oplus(\mathbb{Z}/2)^2$ *or* $\mathbb{Z}/8\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2$. We consider $i_{2_*}(j_2\zeta_6) = j'_2\zeta_6$. Since $\zeta_6\in\pm\{\nu_6, 8\iota_9, 2\sigma_9\}$, thus $j'_2\zeta_6\in\pm j'_2\{\nu_6, 8\iota_9, 2\sigma_9\} = \mp\{j'_2, \nu_6, 8\iota_9\}\circ 2\sigma_{10}$. So $\exists\alpha\in\mp\{j'_2, \nu_6, 8\iota_9\}$, such that $j'_2\zeta_6 = 2\alpha\sigma_{10}$, or say $i_{2_*}(j_2\zeta_6) = 2\alpha\sigma_{10}$ is of order 8. Hence, $\pi_{17}(\Sigma^2\mathbb{H}P^2)$ cannot be $\mathbb{Z}/8\oplus(\mathbb{Z}/2)^3$ or $\mathbb{Z}/8\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2$. Then, $\pi_{17}(\Sigma^2\mathbb{H}P^2)\approx\mathbb{Z}/16\oplus(\mathbb{Z}/2)^2$

For $k = 1$, we know that $sk_{20}(F_1) = S^5\vee S^{13}$,we observe the following exact sequence, here we identify $\pi_{16}(F_1)$ with $\pi_{16}(S^5)\oplus\pi_{16}(S^{13})$, and identify $\pi_{15}(F_1)$ with $\pi_{15}(S^5)\oplus\pi_{15}(S^{13})$,

$$\pi_{17}(S^9) \xrightarrow{\partial_{1_{*17}}} \pi_{16}(S^5) \oplus \pi_{16}(S^{13}) \longrightarrow \pi_{16}(\Sigma\mathbb{H}P^2) \longrightarrow \pi_{16}(S^9) \xrightarrow{\partial_{1_{*16}}} \pi_{15}(S^5) \oplus \pi_{15}(S^{13})$$

We need $\mathrm{cok}(\partial_{1_{*17}})$ and $\mathrm{Ker}(\partial_{1_{*16}})$. We observe the commutative diagram,

$$\begin{tikzcd}
\mathbb{Z}/2\{\varepsilon_8\} \oplus \mathbb{Z}/2\{\bar{\nu}_8\} \oplus \mathbb{Z}/2\{\sigma_8\eta_{15}\} \oplus \mathbb{Z}/2\{\Sigma\sigma' \circ \eta_{15}\} \arrow[d, equal] & \mathbb{Z}/8\{\zeta_5\} \oplus \mathbb{Z}/2\{\nu_5\bar{\nu}_8\} \oplus \mathbb{Z}/2\{\nu_5\varepsilon_8\} \arrow[d, equal] \\
\pi_{16}(S^8) \arrow[r, "\nu_{5_*}"] \arrow[d, "\Sigma"', "onto"] & \pi_{16}(S^5) \arrow[d, "into"] \\
\pi_{17}(S^9) \arrow[r, "\partial_{1_{*17}}"] \arrow[d, equal] & \pi_{16}(S^5) \oplus \pi_{16}(S^{13}) \arrow[d, equal] \\
\mathbb{Z}/2\{\varepsilon_9\} \oplus \mathbb{Z}/2\{\bar{\nu}_9\} \oplus \mathbb{Z}/2\{\sigma_9\eta_{16}\} & \mathbb{Z}/8\{\zeta_5\} \oplus \mathbb{Z}/2\{\nu_5\bar{\nu}_8\} \oplus \mathbb{Z}/2\{\nu_5\varepsilon_8\} \oplus \mathbb{Z}/8\{\nu_{13}\}
\end{tikzcd}$$

So, $\mathrm{Im}(\partial_{1_{*17}}) \approx \mathrm{Im}(\nu_{5_*})$. According to *Page* 152 *of* [1], we have $\nu_5\sigma_8\eta_{15} = \nu_5\varepsilon_8$, and according to *Page* 67 *and Page* 70 *of* [1], we have $\nu_5 \circ \Sigma\sigma' \circ \eta_{15} = 0$. So, $\mathrm{Im}(\nu_{5_*}) = \mathbb{Z}/2\{\nu_5\bar{\nu}_8\} \oplus \mathbb{Z}/2\{\nu_5\varepsilon_8\}$. Successively, $\mathrm{cok}(\partial_{1_{*17}}) = \mathbb{Z}/8\{\zeta_5\} \oplus \mathbb{Z}/8\{\nu_{13}\}$. Next we observe the following commutative diagram,

$$\begin{tikzcd}
\pi_{16}(S^9) \arrow[r, "\partial_{1_{*16}}"] \arrow[d, "id"] & \pi_{15}(S^5) \oplus \pi_{15}(S^{13}) \arrow[d, "\phi_{1_*}"', "proj."] \\
\pi_{16}(S^9) \arrow[r, "\nu_{6_*}"] \arrow[d, equal] & \pi_{16}(S^6) \arrow[d, equal] \\
\mathbb{Z}/16\{\sigma_9\} & \mathbb{Z}/8\{\nu_6\sigma_9\} \oplus \mathbb{Z}/2\{\eta_6\mu_7\}
\end{tikzcd}$$

Here, $\Sigma : \pi_{15}(S^5) \to \pi_{16}(S^6)$ is an isomorphism by *Page* 66 *of* [1]; $\phi_{1_*}|_{\pi_{15}(S^5)}$ is an isomorphism and $\mathrm{Im}(\partial_{1_{*16}}) \subseteq \pi_{15}(S^5)$ by the homotopy-commutative diagram and the commutative diagram of the 15-th homotopy groups it induces.

$$\begin{tikzcd}
\Omega S^9 \arrow[r, "\partial_{1_{*16}}"] \arrow[d, "H_2'"'] & J(M_{S^5}, S^8) \arrow[d, "H_2"] \\
\Omega S^{17} \arrow[r] & \Omega S^{14}
\end{tikzcd}$$

Hence, $\mathrm{Ker}(\partial_{1_{*16}}) = \mathrm{Ker}(\nu_{6_*}) = \mathbb{Z}/2\{8\sigma_9\}$. Then we get the exact sequence,

$$0 \to \mathbb{Z}/8 \oplus \mathbb{Z}/8 \longrightarrow \pi_{16}(\Sigma\mathbb{H}P^2) \longrightarrow \mathbb{Z}/2 \to 0$$

This gives $\pi_{16}(\Sigma\mathbb{H}P^2) \approx \mathbb{Z}/8\oplus\mathbb{Z}/8\oplus\mathbb{Z}/2$ *or* $\mathbb{Z}/16\oplus\mathbb{Z}/8$. Comparing with $\pi_{17}(\Sigma^2\mathbb{H}P^2)$ by the commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/8\oplus\mathbb{Z}/8 \arrow[r] \arrow[d] & \pi_{16}(\Sigma\mathbb{H}P^2) \arrow[r] \arrow[d, "w_{1+1_*}"] & \mathbb{Z}/2 \arrow[r] \arrow[d, "\approx"] & 0 \\ 0 \arrow[r] & \mathbb{Z}/8\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[r] & \mathbb{Z}/16\oplus(\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

So $\exists y\in\pi_{16}(\Sigma\mathbb{H}P^2)$ such that $ord(w_{1+1_*}(y)) = 16$, thus $ord(y)\geq 16$. Hence,$\pi_{16}(\Sigma\mathbb{H}P^2)\approx \mathbb{Z}/16\oplus\mathbb{Z}/8$.

For $k = 0$, $\pi_{15}(\mathbb{H}P^2)\approx\pi_{14}(S^3)\approx\mathbb{Z}/4\oplus(\mathbb{Z}/2)^2$. □

5. A Classification Theorem of A Kind of 3 localized 3-cell Complexes

Lemma 5.1 After localization at 3, let B be a simply connected cell complex of dimension $m\geq 2$ and $\pi_{n-1}(B) = \mathbb{Z}/3\{\gamma\}$ $(n > m)$. Suppose $A = B\cup_g e^n$, and $\widetilde{H}_*(A)$ has notrivial Steenord operation P^1_* of dimension n, then A is the cofibre of γ up to homotopy.

Proof.

We assert that $[g]$ is a generator of $\pi_{n-1}(B)$. To obtain a contradiction, assume $[g]$=0. Then $A\simeq B\vee S^n, \widetilde{H}_*(A) = \widetilde{H}_*(B)\oplus\widetilde{H}_*(S^n)$. So $P^1_*(u)$=0 for any $u\in H_n(A)$. This forces $[g]\neq 0$. Hence the result follows. □

In next part, after localization at 3, we will use $i: S^5\hookrightarrow S^5\vee S^9$, $j: S^9\hookrightarrow S^5\vee S^9$ to denote the inclusions, and $a = \alpha_2(5), b = \alpha_1(7)$ denote the generators of $\pi_{12}(S^5)$ and $\pi_{12}(S^9)$ respectively .

Lemma 5.2 After localization at 3, for all $k\geq 0$, the suspension homomorphisms $\Sigma: \pi_{12+k}(S^{5+k}\vee S^{9+k})\to\pi_{13+k}(S^{6+k}\vee S^{10+k})$ are all isomorphisms, and

$$\pi_{12+k}(S^{5+k}\vee S^{9+k}) = \mathbb{Z}/3\{\Sigma^k(ia)\}\oplus\mathbb{Z}/3\{\Sigma^k(jb)\}.$$

Proof

We only need to show that $\pi_{12+k}(S^{5+k}\vee S^{9+k}) = \mathbb{Z}/3\{\Sigma^k(ia)\}\oplus\mathbb{Z}/3\{\Sigma^k(jb)\}$.

Since $\pi_{12+k}(S^{5+k})\approx\mathbb{Z}/3$ for any $k\geq 0$. By Lemma 2.4.7 , we know $\Sigma: \pi_{12+k}(S^{5+k})\to \pi_{13+k}(S^{6+k})$ $(k\geq 0)$ are all isomorphisms. And $\pi_{12+k}(S^{9+k})\xrightarrow{\Sigma}\pi_{13+k}(S^{10+k})$ are clearly isomorphisms for they are in the stable range. Next, by calculation

$$\begin{aligned}
&\pi_{12+k}(S^{5+k}\vee S^{9+k})\\
&\approx \pi_{11+k}(\Omega(S^{5+k}\vee S^{9+k}))\\
&\approx \pi_{11+k}(\Omega S^{5+k}\times \Omega(\bigvee_{n\geq 0} S^{9+k}\wedge S^{n(4+k)}))\\
&\approx \pi_{12+k}(\ S^{5+k})\oplus \pi_{12+k}(S^{9+k})\\
&\approx \mathbb{Z}/3\oplus\mathbb{Z}/3.
\end{aligned}$$

Therefore, $\pi_{12+k}(S^{5+k}\vee S^{9+k}) = \mathbb{Z}/3\{\Sigma^k(ia)\}\oplus\mathbb{Z}/3\{\Sigma^k(jb)\}$. $\square$

After localization at 3, recall that $i : S^5 \hookrightarrow S^5\vee S^9$, $j : S^9 \hookrightarrow S^5\vee S^9$ are the inclusions, and $a = \alpha_2(5), b = \alpha_1(7)$ are the generators of $\pi_{12}(S^5)$ and $\pi_{12}(S^9)$ respectively, and $h : S^{11} \to \mathbb{H}P^2$ is the attaching map of $\mathbb{H}P^3$. Noticed that we known $\pi_{11+k}(\Sigma^k\mathbb{H}P^2) = \mathbb{Z}/9\{\Sigma^k h\}$, for $k\geqslant 1$ but $k\neq 4$, as well as $\pi_{15}(\Sigma^4\mathbb{H}P^2) = \mathbb{Z}/9\{\Sigma^4 h\}\oplus\mathbb{Z}_{(3)}\{u\}$ where $u = j_4'[\iota_8,\iota_8]$. Let $\mathbb{N} = \mathbb{Z}_+\cup\{0\}$, the set of natural numbers, and we say two 1-connected CW complexes X, Y owing the same homology type if $\widetilde{H}_*(X)\approx\widetilde{H}_*(Y)$ as graded groups. We have

Theorem 5.3 After localization at 3, suppose $A = S^5\cup_a e^{13}$, $c_k = \Sigma^{k-1}(ia+jb)$, then

(**i**) for $k\geqslant 1$ and $k\neq 4$, the 1-connected 3-local CW complexes owing the same homology type as $\Sigma^k\mathbb{H}P^3$ can be classified as

$\Sigma^k\mathbb{H}P^3$, $\Sigma^k\mathbb{H}P^2\cup_{3\Sigma^k h} e^{12+k}$, $\Sigma^k\mathbb{H}P^2\vee S^{12+k}$,

$\Sigma^{k-1}A\vee S^{8+k}, S^{4+k}\vee\Sigma^{4+k}\mathbb{H}P^2$, $(S^{4+k}\vee S^{8+k})\cup_{c_k} e^{12+k}$ and $S^{4+k}\vee S^{8+k}\vee S^{12+k}$ up to homotopy.

(**ii**) the 1-connected CW complexes owing the same homology type as $\Sigma^4\mathbb{H}P^3$ can be classified as

$\Sigma^4\mathbb{H}P^3$, $\Sigma^4\mathbb{H}P^2\cup_{3\Sigma^4 h} e^{16}$, $\Sigma^4\mathbb{H}P^2\vee S^{16}$,

$C_{3^r u}$, (r runs over $\mathbb{N}$), $C_{3^r u+\Sigma^4 h}$, (r runs over $\mathbb{N}$), $C_{3^r u+3\Sigma^4 h}$, (r runs over $\mathbb{N}$),

$\Sigma^3 A\vee S^{12}, S^8\vee\Sigma^8\mathbb{H}P^2$, $(S^8\vee S^{12})\cup_{c_4} e^{16}$ and $S^8\vee S^{12}\vee S^{16}$ up to homotopy.

Proof

(**i**) Notice that $\widetilde{H}_*(\Sigma^k\mathbb{H}P^2\cup_{3\Sigma^k h} e^{12+k})$ has trivial Steenord operation P^1_* of dimension $12+k$ by observing the space $(\Sigma^k\mathbb{H}P^2\cup_{3\Sigma^k h} e^{12+k})/\Sigma^k\mathbb{H}P^1\simeq S^{8+k}\vee S^{12+k}$. And $\widetilde{H}_*((S^{4+k}\vee S^{8+k})\cup_{c_k} e^{12+k})$ has nontrivial Steenord operation P^1_* of dimension $12+k$ by observing the space $(S^{4+k}\vee S^{8+k})\cup_{c_k} e^{12+k})/S^{4+k}\simeq\Sigma^{k+4}\mathbb{H}P^2$. And $\pi_{11+k}(\Sigma^{k-1}A\vee S^{8+k}) = \mathbb{Z}/3$, but $\pi_{11+k}(S^{4+k}\vee S^{8+k}\vee S^{12+k}) = (\mathbb{Z}/3)^2$. We need only show that

$$\Sigma^k\mathbb{H}P^2 \cup_{3\Sigma^k h} e^{12+k} \text{ and } \Sigma^k\mathbb{H}P^2 \vee S^{12+k}$$

are not homotopy equivalent , and

$$S^{4+k} \vee \Sigma^{4+k}\mathbb{H}P^2 \text{ and } (S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k}$$

are also not homotopy equivalent.

As a matter of fact, we have the homotopy-commutative diagram,

$$\begin{tikzcd} S^{11+k} \arrow[r, "3\Sigma^k h"] \arrow[d] & \Sigma^k\mathbb{H}P^2 \arrow[d] \\ D^{12+k} \arrow[r] & \Sigma^k\mathbb{H}P^2 \cup_{3\Sigma^k h} e^{12+k} \end{tikzcd}$$

Since $\pi_{11+k}(\Sigma^k\mathbb{H}P^2) = \mathbb{Z}/9\{\Sigma^k h\}$, this diagram implies $|\pi_{11+k}(\Sigma^k\mathbb{H}P^2 \cup_{3\Sigma^k h} e^{12+k})| \leq 7$, while $\pi_{11+k}(\Sigma^k\mathbb{H}P^2 \vee S^{12+k}) \approx \mathbb{Z}/9$. Thus $\Sigma^k\mathbb{H}P^2 \cup_{3\Sigma^k h} e^{12+k}$ and $\Sigma^k\mathbb{H}P^2 \vee S^{12+k}$ are not homotopy equivalent.

Next, to obtain a contradiction , suppose $S^{4+k} \vee \Sigma^{4+k}\mathbb{H}P^2 \simeq (S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k}$. Then, $X := (S^{4+k} \vee \Sigma^{4+k}\mathbb{H}P^2)/S^{8+k} \simeq Y := ((S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k})/S^{8+k}$. Since $X = S^{4+k} \vee S^{12+k}$, and $Y \simeq S^{4+k} \cup_{\Sigma^{k-1}a} e^{12+k}$, we notice that $\Sigma^{k-1}a$ is a generator of $\pi_{11+k}(S^{4+k})$, so, $\pi_{11+k}(X) = \mathbb{Z}/3$, $\pi_{11+k}(Y) = 0$, this forces that $S^{4+k} \vee \Sigma^{4+k}\mathbb{H}P^2$ and $(S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k}$ are not homotopy equivalent.

(ii) We define $\langle\alpha\rangle = \{x\alpha \mid x \in \mathbb{Z}_{(3)}\}$. As a matter of fact, since $\pi_{15}(\Sigma^4\mathbb{H}P^2) = \mathbb{Z}/9\{\Sigma^4 h\} \oplus \mathbb{Z}_{(3)}\{u\}$ where $u = j_4' \circ [\iota_8, \iota_8]$. For any $[f] \in \pi_{15}(\Sigma^4\mathbb{H}P^2)$, by observing the following homotopy-commutative diagram (see lemma 2.7.9) with the second row fibre sequence where $sk_{22}(J(M_{\Sigma^4\mathbb{H}P^2}, S^{15}) = J_1(M_{\Sigma^4\mathbb{H}P^2}, S^{15}) \simeq \Sigma^4\mathbb{H}P^2$ and by observing the commutative diagram of homotopy groups it induced,

$$\begin{tikzcd} S^{15} \arrow[r, "f"] \arrow[d, "\Omega\Sigma"] & \Sigma^4\mathbb{H}P^2 \arrow[d, "into"] & & \\ \Omega S^{16} \arrow[r] & J(M_{\Sigma^4\mathbb{H}P^2}, S^{15}) \arrow[r] & C_f \arrow[r] & S^{16} \end{tikzcd}$$

it's easy to get that

$$\pi_{15}(\Sigma^4\mathbb{H}P^3) \approx \frac{\mathbb{Z}/9\{\Sigma^4 h\} \oplus \mathbb{Z}_{(3)}\{u\}}{\langle\Sigma^4 h\rangle} \approx \mathbb{Z}_{(3)},$$

$\pi_{15}(\Sigma^4\mathbb{H}P^2 \cup_{3\Sigma^4 h} e^{16}) \approx \frac{\mathbb{Z}/9\{\Sigma^4 h\}\oplus\mathbb{Z}_{(3)}\{u\}}{\langle 3\Sigma^4 h\rangle} \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/3.$

$\pi_{15}(\Sigma^4\mathbb{H}P^2 \vee S^{16}) \approx \mathbb{Z}_{(3)} \oplus \mathbb{Z}/9.$

$\pi_{15}(C_{3^t u}) \approx \frac{\mathbb{Z}/9\{\Sigma^4 h\}\oplus\mathbb{Z}_{(3)}\{u\}}{\langle 3^t u\rangle} \approx \mathbb{Z}/9 \oplus \mathbb{Z}/3^t, (t \in \mathbb{N}),$

$\pi_{15}(C_{3^t u+\Sigma^4 h}) \approx \frac{\mathbb{Z}/9\{\Sigma^4 h\}\oplus\mathbb{Z}_{(3)}\{u\}}{\langle 3^t u+\Sigma^4 h\rangle} \approx \mathbb{Z}/3^{t+2}, (t \in \mathbb{N})$

$$\pi_{15}(C_{3^t u+3\Sigma^4 h}) \approx \frac{\mathbb{Z}/9\{\Sigma^4 h\}\oplus\mathbb{Z}_{(3)}\{u\}}{\langle 3^t u+3\Sigma^4 h\rangle} \approx \begin{cases} \mathbb{Z}/3^{t+1} \oplus \mathbb{Z}/3, & t \in \mathbb{Z}_+, \\ \mathbb{Z}/9, & t = 0 \end{cases}$$

Since $\Sigma C_{3^r u} = \Sigma^5\mathbb{H}P^2 \vee S^{17}, \Sigma C_{3^r u+\Sigma^4 h} = \Sigma^5\mathbb{H}P^3, \Sigma C_{3^r u+3\Sigma^4 h} = \Sigma^5\mathbb{H}P^2 \cup_{3\Sigma^k h} e^{17}$, by (**i**), the theorem is established. □

Theorem 5.4 After localization at 3, for a 1-connected CW complex X owing the same homology type as $\Sigma^k\mathbb{H}P^3$ ($k \geqslant 1$ and $k \neq 4$), if $\widetilde{H}_*(X)$ has nontrivial Steenord operations P^1_* of dimension $8+k$ and $12+k$, then $X \simeq \Sigma^k\mathbb{H}P^3$.

Proof.

Since the given, $\widetilde{H}_*(X) = \mathbb{Z}/3\{x, y, z\}$ ($|x| = 4+k, |y| = 8+k, |z| = 12+k$), with the Steenord operations $P^1_*(z) = y$, $P^1_*(y) = x$. Firstly we observe $sk_{8+k}(X)$, it is obviously of type of $S^{4+k} \cup e^{8+k}$. Since $\pi_{7+k}(S^{4+k}) = \pi^s_3(S^0) = \mathbb{Z}/3$, and notrivial Steenord operation $P^1_*(y) = x$, hence $sk_{8+k}(X) \simeq \Sigma^k\mathbb{H}P^2$ localized at 3 by Lemma 4.1. Now we could set $X = \Sigma^k\mathbb{H}P^2 \cup_f e^{12+k}$. Recall that we have known $\pi_{11+k}(\Sigma^k\mathbb{H}P^2) = \mathbb{Z}/9\{[\Sigma^k h]\}$ ($k \geq 1$ and $k \neq 4$), where h is the attaching map of e^8 of $\mathbb{H}P^2$. We assert that $[f]$ is a generator of $\pi_{11+k}(\Sigma^k\mathbb{H}P^2)$.

To obtain a contradiction, suppose $[f] = \pm 3[\Sigma^k h]$ or 0. We consider the space $X/\Sigma^k\mathbb{H}P^1 = S^{8+k} \cup_{q\circ f} e^{12+k}$ where q is the pinch map $\Sigma^k\mathbb{H}P^2 \to X/\Sigma^k\mathbb{H}P^1$. If $[f] = 0$, we have $[q\circ f] = 0$. If $[f] = \pm 3[\Sigma^k h]$, we also have $[q \circ f] = 0$ because $[q\circ f] \in \pi_{11+k}(S^{8+k}) = \pi^s_3(S^0) = \mathbb{Z}/3$. Then $X/\Sigma^k\mathbb{H}P^1 \simeq S^{8+k} \vee S^{12+k}$. However $\widetilde{H}_*(X/\Sigma^k\mathbb{H}P^1) = \mathbb{Z}/3\{y', z'\}$ ($|y'| = 8+k, |z'| = 12+k$) with $P^1_*(z') = y'$, while $\widetilde{H}_*(S^{8+k} \vee S^{12+k})$ only has trivial Steenord operation. It forces $[f]$ is a generator of $\pi_{11+k}(\Sigma^k\mathbb{H}P^2)$. Then $X \simeq \Sigma^k\mathbb{H}P^3$ localized at 3.

□

Note 5.5 The theorem above would be failed if allowing $k = 4$. A counter example is, $C_{u+\Sigma^4 h} = S^8 \cup e^{12} \cup e^{16}$ has nontrivial Steenord operations P^1_* of dimension 12 and 16, but $C_{u+\Sigma^4 h}$ and $\Sigma^4\mathbb{H}P^3$ do not have the same homotopy type.

From now on, $a \otimes b$ is written as ab for short. Recall the statements in 2.12, and one could see more details on these in [8], we have

Theorem 5.6 After localization at 3,

$$\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2 \simeq S^{13} \vee \Sigma^5\mathbb{H}P^3.$$

Proof.

Firstly we have $\widetilde{H}_*(\mathbb{H}P^2)$=$\mathbb{Z}/3\{x, y\} := V$, where $|x| = 4, |y| = 8$. Then $\widetilde{H}_*(\mathbb{H}P^2 \wedge \mathbb{H}P^2) = V\otimes V = \mathbb{Z}/3\{xx, xy, yx, yy\}$. Choose $u = \frac{1+(12)}{2}$, $v = \frac{1-(12)}{2} \in \mathbb{Z}_{(3)}[S_2]$, therefore, $1 = u + v$ is an orthogonal decomposition of the identity in $\mathbb{Z}_{(3)}[S_2]$, any element in $\mathbb{Z}_{(3)}[S_2]$ decides a self map of $\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^3$, we have

$$\Sigma\mathbb{H}P^2\wedge \mathbb{H}P^2 \simeq \mathrm{hocolim}_u(\Sigma\mathbb{H}P^2 \wedge \mathbb{H}P^2) \vee \mathrm{hocolim}_v(\Sigma\mathbb{H}P^2 \wedge \mathbb{H}P^2),$$

Since the homotopy colimits commute with the homology functor, we have,
$\widetilde{H}_*(\mathrm{hocolim}_u(\Sigma\mathbb{H}P^2 \wedge \mathbb{H}P^2))$=$\mathrm{Im}(u_*)$=$\mathbb{Z}/3\{\sigma(xy - yx)\}$ where σ are the suspension isomorphisms $\widetilde{H}_*(-) \to \widetilde{H}_{*+1}(\Sigma -)$ of the homology groups, and $|\sigma(xy - yx)| = 13$. Then,
$\mathrm{hocolim}_u(\Sigma\mathbb{H}P^3 \wedge \mathbb{H}P^3) \simeq S^{13}$.
$\widetilde{H}_*(\mathrm{hocolim}_v(\Sigma\mathbb{H}P^2 \wedge \mathbb{H}P^2))$=$\mathrm{Im}(v_*)$=$\mathbb{Z}/3\{\sigma(xy + yx), \sigma(xx), \sigma(yy)\}$. The Steenord operations are given by $P^1_*(\sigma(yy))$=$\sigma(xy + yx)$, $P^2_*(\sigma(yy))$=$\sigma(xx)$, $P^1_*(\sigma(xy + yx))$=$\sigma(xx)$. By theorem 5.4, we get

$$\mathrm{hocolim}_v(\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2) \simeq \Sigma^5\mathbb{H}P^3.$$ □

Theorem 5.7 After localization at 3,

$$\Sigma\mathbb{H}P^3\wedge\mathbb{H}P^3 \simeq \Sigma^9\mathbb{H}P^3\vee Y,$$

where Y is a 6-cell complex with $sk_{13}(Y) = \Sigma^5\mathbb{H}P^2$.

Proof. $\widetilde{H}_*(\mathbb{H}P^3)$=$\mathbb{Z}/3\{x, y, z\} := V$, $P^1_*(z) = -y$, $P^2_*(z) = x$, $P^1_*(y) = x$, where $|x|$=4,$|y|$=8 and $|z|$=12. And $\widetilde{H}_*(\mathbb{H}P^3\wedge\mathbb{H}P^3)$= $V \otimes V$=$\mathbb{Z}/3\{xx, xy, xz, yx, yy, yz, zx, zy, zz\}$.
Choose u=$\frac{1+(12)}{2}$,v=$\frac{1-(12)}{2}$$\in \mathbb{Z}_{(3)}[S_2]$, therefore, $1 = u + v$ is an orthogonal decomposition of

the identity in $\mathbb{Z}_{(3)}[S_2]$. Thus,

$$\Sigma\mathbb{H}P^3 \wedge \mathbb{H}P^3 \simeq \mathrm{hocolim}_u(\Sigma\mathbb{H}P^3 \wedge \mathbb{H}P^3) \bigvee \mathrm{hocolim}_v(\Sigma\mathbb{H}P^3 \wedge \mathbb{H}P^3),$$

Since the homotopy colimits commute with the homology functor, we have, $\widetilde{H}_*(\mathrm{hocolim}_v\Sigma\mathbb{H}P^3 \wedge \mathbb{H}P^3)$=Im$(v_*)$=$\mathbb{Z}/3\{\sigma(xy-yx), \sigma(xz-zx), \sigma(zy-yz)\}$,where $|\sigma(xy-yx)| = 13, |\sigma(xz-zx)| = 17, |\sigma(zy-yz)| = 21$ and σ are the suspension isomorphisms $\widetilde{H}_*(-) \to \widetilde{H}_{*+1}(\Sigma -)$ of the homology groups. There are nontrivial Steenord operations $P^1_*(\sigma(zy-yz))$=$-\sigma(xz-zx)$, $P^1_*(\sigma(xz-zx))$=$\sigma(yx-xy)$. By theorem 5.4 , we get

$$\mathrm{hocolim}_v(\Sigma\mathbb{H}P^3 \wedge \mathbb{H}P^3) \simeq \Sigma^9\mathbb{H}P^3.$$

Let Y=hocolim$_u(\Sigma\mathbb{H}P^3 \wedge \mathbb{H}P^3)$, then

$$\widetilde{H}_*(Y)=\mathrm{Im}(u_*)=\mathbb{Z}/3\{\sigma(xy+yx), \sigma(xz+zx), \sigma(zy+yz), \sigma(xx), \sigma(yy), \sigma(zz)\}.$$

Here, the dimensions of the basis $\sigma(xy+yx), \sigma(xz+zx), \sigma(zy+yz), \sigma(xx), \sigma(yy), \sigma(zz)$ are 13 ,17 ,21 ,8 ,16 ,24 respectively. And $P^1_*(\sigma(xy+yx)) = -\sigma(xx)$, hence, $sk_{13}(Y) = \Sigma^5\mathbb{H}P^2$. Then the result follows. □

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